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Fixed Point Theorems Using Expansive Mappings in Partial Metric Spaces

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Abstract

This paper presents new fixed-point theorems for expansive mappings in partial metric spaces, extending classical results from metric spaces to a more general framework. Partial metric spaces, introduced by Matthews, allow nonzero self-distances, thereby providing a richer structure suitable for modeling computational and topological phenomena. By considering three self-mappings $S, T,$and Uon a complete partial metric space and introducing a novel contraction condition, we establish the existence and uniqueness of a common fixed point. The results not only generalize several well-known fixed-point theorems but also demonstrate the flexibility of partial metric spaces in accommodating non-commuting mappings. These findings contribute to the growing intersection of fixed-point theory, topology, and theoretical computer science, enhancing its applicability in computational models and analysis.

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1. Introduction

In this paper, I investigated and generalized some well-known results on common fixed points to the class of partial metric spaces. Many problems in pure and applied mathematics reduce to finding a common fixed point of self-mapping operators defined on metric spaces. Partial metric spaces generalize traditional metric spaces by allowing the self-distance of points to be non-zero while still satisfying the properties of symmetry and a modified version of the triangle inequality. This study extends and analyzes these results within the framework of partial metric spaces, offering broader applicability and deeper insights into fixed point theory.

Partial metric spaces, introduced by Matthews [1, 2], generalize classical metric spaces by relaxing the axiom $d(x,x)=0$ to $d(x,x)\leq d(x,y)$. This modification enables the modeling of computational processes where self-distance (e.g., computational complexity or information content) need not vanish, making partial metrics particularly valuable in computer science applications such as program semantics, domain theory, and logic programming [3–7]. Matthews [2] established foundational results for these spaces, including convergence properties and fixed-point theorems for contractive mappings. Specifically, he proved that any contraction T on a complete partial metric space (X,p), satisfying $p(Tx,Ty)\leq kp(x,y)$ for $0\leq k<1$, admits a unique fixed point. These results have spurred extensive research, with recent works [8–16] extending classical fixed-point theorems, originally formulated for metric spaces to the partial metric framework.

The versatility of partial metric spaces lies in their ability to encode both metric and topological structures simultaneously. For instance, they generate T_0 topologies and induce equivalent metrics (e.g., dp and dm) that recover classical results while accommodating richer computational and geometric interpretations.

This paper extends common fixed-point theorems for pairs of self-mappings (both non-commuting and commuting) from metric spaces [17] to partial metric spaces, addressing existence, uniqueness, and convergence under weakened assumptions.

2. Preliminaries

Definition 1.1 (Partial Metric Space [1])

A partial metric space (PMS) is a pair (X, p), where X is a nonempty set and $p: X \times X \to R0+$ satisfies:

1. **Nonzero Self-Distance**: $x = y \Leftrightarrow p(x,x) = p(y,y) = p(x,y)$,
2. **Small Self-Distance**: $p(x,x) \leq p(x,y)$,
3. **Symmetry**: $p(x,y) = p(y,x)$,
4. **Modified Triangle Inequality**: $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Induced Metrics

For a PMS (X,p), the functions $dp, dm: X \times X \to R0+$, defined as:

$$dp(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

$$dm(x,y) = max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\},$$

are equivalent metrics on X.

Example 1.2 [1, 2]

Let $X = [a,b]: a, b \in R, a \leq b$ and define $p([a,b],[c,d]) = max\{b,d\} - min\{a,c\}$. Then (X,p) is a PMS.

Topological Concepts in PMS

1. **Convergence**: A sequence xn in (X,p) converges to $x \in X$ if $p(x,x) = \lim_{n\to\infty} p(x, x_n)$.
2. **Cauchy Sequence**: xn is Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
3. **Completeness**: (X,p) is complete if every Cauchy sequence xn converges (under τ_p) to some $x \in X$ with $\lim_{n,m\to\infty} p(x_n, x_m)$.
4. **Continuity**: A mapping $f: X \to X$ is continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $f\left(B_p(x_0, \delta)\right) \subseteq B_p(f(x_0), \varepsilon)$.

Key Lemmas

1. **Lemma 1.4** [1, 2, 13]:
   a) xn is Cauchy in $(X,p) \Leftrightarrow xn$ is Cauchy in (X, dp).
   b) (X,p) is complete $\Leftrightarrow (X, dp)$ is complete. Moreover,

$$\lim_{n\to\infty} d_p(x, x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m)$$

2. **Lemma 1.6** [15]: If $xn \to z$ in (X,p) with $p(z,z) = 0$, then $\lim_{n\to\infty} p(x_n, y) = p(z,y)$ for all $y \in X$.

Remark 1.5

In a PMS (X,p):

A. $p(x,y) = 0 \Longrightarrow x = y$;

B. $x \neq y \Longrightarrow p(x,y) > 0$.

These preliminaries establish the framework for extending fixed point theory to partial metric spaces, bridging classical analysis with applications in computation and topology.

3. Main Results

Theorem 3.2: Let (X,p) be a complete partial metric space, and $S, T, U: X \to X$ be three self- mappings. Suppose there exists $r \in [0,1)$ such that for all $x, y, z \in X$:

$$p(Sx, Ty) \leq r \max\{ p(Ux, x),\ p(Uy, y),\ p(x,y),\ \tfrac{1}{2}\, p(Sx, y) + p(Ty, x)$$

Then, there exists a unique $z \in X$ such that $Sz = Tz = Uz = z$.

Proof

Let $x0 \in X$ an arbitrary initial point. the sequence x_n cyclically as follows:

$$x_1 = S_{x_0},$$
$$x_2 = T_{x_1},$$
$$x_3 = U_{x_2},$$
$$x_4 = S_{x_3},$$
$$x_5 = T_{x_4},$$
$$x_6 = U_{x_5},$$
$$\vdots$$

In general, for $k \geq 0$:

$$x_{3k+1} = Sx_{3k}$$
$$x_{3k+2} = Tx_{3k+1}$$
$$x_{3k+3} = Ux_{3k+2}, k \geq 0$$

We analyze the sequence x_n using the given contraction condition.

Case 1: For $n = 3k$, consider $p(x_{3k+1}, x_{3k+2}) = p(Sx_{3k}, Tx_{3k+1})$
Applying the contraction condition

$$p(Sx_{3k}, Tx_{3k+1}) \leq r \max\left\{ p(Ux_{3k}, x_{3k}), p(Ux_{3k+1}, x_{3k+1}), p(x_{3k}, x_{3k+1}), \left(\frac{1}{2}\right)[p(Sx_{3k}, x_{3k+1}) + p(Tx_{3k+1}, x_{3k})]\right\}$$

Simplify using properties of p:

$$p(Ux_{3k}, x_{3k}) = p(x_{3k+3}, x_{3k})(since\ x_{3k+3} = Ux_{3k+2})$$

$$p(Sx_{3k}, x_{3k+1}) = p(x_{3k+1}, x_{3k+1}) \leq p(x_{3k+1}, x_{3k})\ by\ (PM2)$$

The dominant term reduces to $p(x_{3k}, x_{3k+1})$ Thus:

$$p(x_{3k+1}, x_{3k+2}) \leq r \cdot p(x_{3k}, x_{3k+1})$$

Case 2: For $n = 3k + 1$ and $n = 3k + 2$, analogous inequalities hold:

$$p\left(x_{\{3k+2\}}, x_{\{3k+3\}}\right) \leq r, p\left(x_{\{3k+1\}}, x_{\{3k+2\}}\right),$$

$$p\left(x_{\{3k+3\}}, x_{\{3k+4\}}\right) \leq r, p\left(x_{\{3k+2\}}, x_{\{3k+3\}}\right)$$

From the above inequalities, for all $n \geq 0$:

$$p(x_{n+1}, x_n) \leq r^n \cdot p(x_0, x_1)$$

For the induced metric $dp(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ we have:

$$d_{p(x_n, x_{n+1})} \leq 2p(x_n, x_{n+1}) \leq 2r^n \cdot p(x_0, x_1)$$

For $m>n$, by the triangle inequality:

$$d_{p(x_n, x_m)} \leq \sum_{k=n}^{m-1} d_{p(x_k, x_{k+1})} \leq 2p(x_0, x_1) \sum_{k=n}^{\infty} r^k$$

Since $r \in [0,1)$, the series $\sum_{k=n}^{\infty} r^k$ converges to $\frac{r^n}{1-r}$. Therefore, $dp(x_n, x_m) \to 0$ as $n \to \infty$, proving x_n is Cauchy in (X, dp). By Lemma 1.4, (X, p) is complete, so x_n converges to some $z \in X$ in dp. Moreover

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0$$

For *S*: Apply the contraction condition to $p(S_z, T_z)$:

$$p(Sz, Tz) \leq r \max\left\{ p(Uz, z), p(Tz, Tz), p(z, z), \left(\frac{1}{2}\right)[p(Sz, z) + p(Tz, Sz)]\right\}$$

Since, $p(z, z) = 0$ and $p(Sz, z) = p(Tz, z) = 0$ (by convergence), this reduces to:

$$p(Sz, Tz) \leq r \cdot p(Sz, Tz) \Longrightarrow p(Sz, Tz) = 0 \Longrightarrow Sz = Tz.$$

For *T* and *U*: Similarly, apply the contraction condition to $p(Tz, Uz)$ and $p(Uz, Sz)$:

$$p(Tz, Uz) \leq r \max\left\{ p(Uz, z), p(U(Uz), Uz), p(z, Uz), \left(\frac{1}{2}\right)[p(Tz, z) + p(Uz, Uz)]\right\} = 0,$$

$$p(Uz, Sz) \leq r \max\left\{ p(Uz, z), p(Sz, Sz), p(Uz, Sz), \left(\frac{1}{2}\right)[p(Uz, Sz) + p(Sz, Uz)]\right\} = 0.$$

Thus, $Tz = Uz = Sz = z$.

Uniqueness: Suppose *w* is another common fixed point. Then:

$$p(z, w) = p(Sz, Tw) \leq r \max\left\{ p(Uz, z), p(Uw, w), p(z, w), \left(\frac{1}{2}\right)[p(Sz, w) + p(Tw, z)]\right\} = r\, p(z, w)$$

Since $r < 1$, $p(z, w) = 0 \; implies \; z = w$
This completes the proof

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