International Journal of Applied Mathematics and Numerical Research

On admission of M_0-completion by non-isomorphic digraphs and their associated partial matrices

Grace Oduor [1]*, Benard Okelo [2]
[1-2] Department of Pure and Applied Mathematics, Jaramogi Oginga Odinga University of Science and Technology, P. O. Box 210-40601, Bondo, Kenya

* Corresponding Author: **Grace Oduor**

Article Info

Impact Factor (RSIF): 8.62
ISSN (Online): 3107-7110
Volume: 02
Issue: 03
Received: 21-02-2026
Accepted: 23-03-2026
Published: 25-04-2026
Page No: 01-27

Abstract
Matrices are very important ingredients studies involving in linear algebra and matrix theory. The elements of a matrix can be manipulated through addition and multiplication or even be decomposed in various ways to decipher simple and intricate concepts. The matrix completion problem concentrates on interrogating and determining whether or not a completion of a partial matrix exists within a certain cluster of matrices. A desired type is arrived at by outlining descriptive characteristics and then choices for the unspecified entries made from the same set so that the matrix thereof is the desired type. Let γ denote a specified class of matrices, a pattern is said to admit γ completion if every partial γ – matrix consistent with the pattern can be completed to a full matrix in γ. In this paper, we investigate non-isomorphic digraphs with five vertices and four arcs. We have established that all associated partial matrices that are not cycles admits M_0-completion whereas those corresponding to cyclic digraphs do not.

Keywords: Digraph, Cyclic, Partial matrix, M_0-completion

1. Introduction

Matrices are an essential life line in linear algebra and matrix theory. The elements of a matrix can be manipulated through addition and multiplication or even be decomposed in various ways to decipher simple and intricate concepts.
The matrix completion problem concentrates on interrogating and determining whether or not a completion of a partial matrix exists within a certain cluster of matrices. A desired type is arrived at by outlining descriptive characteristics and then choices for the unspecified entries made from the same set so that the matrix thereof is the desired type. Let γ denote a specified class of matrices, a pattern is said to admit γ completion if every partial γ – matrix consistent with the pattern can be completed to a full matrix in γ.
Over the years matrix completion problems have been studied using graphs and digraphs. Extensively, graphs have been utilized to research on Patterns that are positionally symmetric like the inverse M-matrices and P-matrices. Digraphs (directed graphs), on the other hand, have been used for patterns without positional symmetry.
Matrix completion problems come into play whenever there are some missing parts of a given set of data but it is known that the full matrix data must have given properties. Completion problems are experienced in a number of applications such as Physics (recovering a quantum state from incomplete measurements), Statistics (entropy methods for missing data), Chemistry (the molecular conformation problems) and equally within matrix theory (e.g. determinant inequalities). Moreover, completion

problems have proffered a valuable mechanism for understanding matrix structure more deeply.

2. Preliminaries

A matrix is a rectangular array of numbers which can be real or complex. The numbers in the array are its elements or entries. If the matrix has m rows and n columns it is an $m \times n$ matrix. If $m = n$ then it is a square matrix.

A partial matrix is a matrix in which some entries are while the other remaining entries are free to be chosen. A completion of a partial matrix is an assignment of values to all the unspecified entries. A pattern for an $n \times n$ matrix is a matrix is a set of index positions, that is, a subset {1,…,n}×{1,…,n}. A partial matrix is said to specify a pattern if its specified entries occur precisely at the positions listed in the pattern.

A sub-matrix of a matrix A is any matrix obtained by selecting a subset of its srows and columns.For a subset α of $(1, 2, ..., n)$, the principal sub-matrix A(α), is obtained by deleting all rows and columns whose indices are not in α. A principal minor is the determinant of a principal sub-matrix.

A real matrix A is called a Q-matrix if for every $k = 1, 2, ..., n$ the sum of all $k \times k$ principal minors is positive.A partial matrix is a partial Q -matrix if all the $k \times k$ principal submatrices that are fully specified are Q-matrices.

A real $n \times n$ matrix is called a P_0-matrix if all $k \times k$ principal minors of A are positive. A partial P_0-matrix is a partial matrix in which all the fully specified principal sub-matrices are P_0-matrices.

A real $n \times n$ matrix is called a M_0-matrix if its off diagonal entries are all non-positive and all its principal minors are non-negative. Thus, an M_0- matrix is a P_0 –matrix whose off –diagonal entries are non-positive. A partial M_0-matrix is a partial matrix in which all fully specified principal sub-matrices are M_0-matrices. A partial M_0-matrix that includes all diagonal entries can be completed to an M_0-matrix if and only if the zero completion (obtained by setting all unspecified off-diagonal entries to 0) is an M_0-matrix.

A graph $G = (V, E)$ consists of a non-empty finite set V of vertices and a set E of unordered pairs (v, u) of vertices called edges. A digraph (directed graph) is a graph whose edges have directions. The edges of a digraph are called arcs. Thus, a digraph $D = (V, E)$ consists of a non -empty finite set V of vertices and a set E of edges. The order of D is the number of vertices of D. A digraph $H = (V, E)$ is sub-digraph of a digraph $D = (V, E)$ if V is a sub-set of $V(D)$ and $E(H)$ is a sub-set of $E(D)$. A complete digraph is a directed graph in which every pair of distinct vertices is connected by all possible directed edges. A complete sub-digraph is called a clique. Two digraphs are said to be isomorphic if they have the same structure and differ only in the labeling of their vertices.

A digraph has M_0-completion if every partial M_0-matrix specifying can be completed to an M_0-matrix. A partial matrix has Zero completion if the matrix can be completed to a γ – matrix by equating all the unspecified entries to zero. A pattern is said to be positionally symmetric if whenever a_{ji} is in the pattern, then a_{ji} is also in the pattern. We adopt the following notations for this work.

The entry d_{ij} denotes a specified diagonal entry, a_{ji} a specified non-diagonal entry and x_{ij} an unspecified non-diagonal entry where $1 \le i, j \le n$. Capital letters denote matrices. $\aleph$ is the set of natural numbers $(1, 2, 3, ..., n)$. $A(\beta)$ denotes the principal sub-matrix where β is a subset of $\aleph$. $|A(\beta)|$ is the determinant of the principal sub-matrix $A(\beta)$.

3. Literature Review

In 1984 Grone *et al* [11] studied the positive definite completions of partial Hermitian matrices. In the same year C. R.1985 Horn and Johnson [6] studied Matrix Analysis. Johnson (1990) [7] outlined a scrutiny of matrix completion problems focusing on the positive definite completion problem. Johnson and Smith (1999) [8] studied the "The symmetric inverse M-completion problem". Hogben (2001) [4] studied "Graph Theoretic Method for matrix completion problems". Choi *et al* (2002) [1] studied "The P_0-Matrix Completion Problem". Hogben *et al* (2003) [3] worked on "The Nonnegative P_0- Matrix Completion" using graph theoretic method. Wangness *et al* (2006) [5] worked on "Completion Problems for various classes of P-Matrices". DeAlba *et al* (2009) [2] worked on "The Q-Matrix Completion Problem".

J. Mutembei (2007) [9] worked on "M_0-Matrix completion problem for digraphs of 2×2 and 3×3 matrices.

R. Kabusia (2008) [10] worked on "The M_0-Matrix Completion problem for digraphs of 4×4 matrices.

This work isolates several non-isomorphic digraphs of order five with four arcs and proves that the non-cyclic digraphs have the M_0-Matrix Completion property, whereas the cyclic digraphs do not.

4. Digraphs that have M_0-Completion

In this chapter, each digraph is considered as a separate case. The digraphs under consideration are of 5×5 matrices with 4 edges, that is, $q = 4$.Throughout this study all digraphs are assumed to contain diagonal positions,and the matrices associated with these digraphs are taken to be partial M_0 -matrices.

The partial M_0 - matrices are drawn from the digraphs in the following chronology. An edge in the digraph signify a specific entry a_{ij} in the partial M_0 -matrix,whereas the absence of edge in the digraph signifies an arbitrary or unspecified entry x_{ij} .Since the digraphs are assumed to include all diagonal positions,the diagonal entries denoted by d_{ij} are all specified. This assumption allows for the application of the lemmas below in establishing the stated results.

Lemma 2.1. A pattern that includes all diagonal positions has P_0 - completion.

Lemma 2.2. Zero Completion Criterion is admitted by Partial M_0 -matrices.

Proof. Let A be a partial partial M_0 -matrix in which all diagonal entries are specified. Then A admits an M_0- completion if and only if its zero completion — obtained by setting all unspecified off-diagonal entries to zero — is an M_0 -matrix.(see Johnston and Smith 1991)

Remark 2.3. Every principal minor of a P_0 -matrix is categorically non negative and this is a prerequisite for M_0 -completion. Moreover, $d_{11} \geq 0, d_{22} \geq 0, d_{33} \geq 0, d_{44} \geq 0, d_{55} \geq 0$.

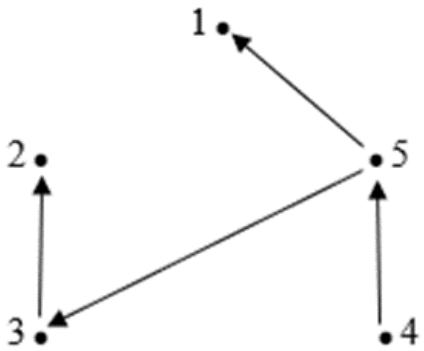

Fig 1:

$$A = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & a_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ a_{51} & x_{52} & a_{53} & x_{54} & d_{55} \end{bmatrix}$$

$$|A(1,2)| = d_{11}d_{22} - x_{12}x_{21}$$
$$|A(1,3)| = d_{11}d_{33} - x_{13}x_{31}$$
$$|A(1,4)| = d_{11}d_{44} - x_{14}x_{41}$$
$$|A(1,5)| = d_{11}d_{55} - x_{15}a_{51}$$
$$|A(2,3)| = d_{22}d_{33} - x_{23}a_{32}$$
$$|A(2,4)| = d_{22}d_{44} - x_{24}x_{42}$$
$$|A(2,5)| = d_{22}d_{55} - x_{25}x_{52}$$
$$|A(3,4)| = d_{33}d_{44} - x_{34}x_{43}$$
$$|A(3,5)| = d_{33}d_{55} - x_{35}a_{53}$$
$$|A(4,5)| = d_{44}d_{55} - a_{45}x_{54}$$

$$\left|A(1,2,3)\right| = d_{11}\left(d_{22}d_{33} - x_{23}a_{32}\right) - x_{12}\left(x_{21}d_{33} - x_{23}x_{31}\right) + x_{13}\left(x_{21}a_{32} - d_{22}x_{31}\right)$$

$$\left|A(1,2,4)\right| = d_{11}\left(d_{22}d_{44} - x_{24}x_{42}\right) - x_{12}\left(x_{21}d_{44} - x_{24}x_{41}\right) + x_{14}\left(x_{21}x_{42} - d_{22}x_{41}\right)$$

$$\left|A(1,2,5)\right| = d_{11}\left(d_{22}d_{55} - x_{25}x_{52}\right) - x_{12}\left(x_{21}d_{55} - x_{25}a_{51}\right) + x_{15}\left(x_{21}x_{52} - d_{22}a_{51}\right)$$

$$\left|A(1,3,4)\right| = d_{11}\left(d_{33}d_{44} - x_{34}x_{43}\right) - x_{13}\left(x_{31}d_{44} - x_{34}x_{41}\right) + x_{14}\left(x_{31}x_{43} - d_{33}x_{41}\right)$$

$$\left|A(1,3,5)\right| = d_{11}\left(d_{33}d_{55} - x_{35}a_{53}\right) - x_{13}\left(x_{31}d_{55} - x_{35}a_{51}\right) + x_{15}\left(x_{31}a_{53} - d_{33}a_{51}\right)$$

$$\left|A(1,4,5)\right| = d_{11}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{14}\left(x_{41}d_{55} - a_{45}a_{51}\right) + x_{15}\left(x_{41}x_{54} - d_{44}a_{51}\right)$$

$$\left|A(2,3,4)\right| = d_{22}\left(d_{33}d_{44} - x_{34}x_{43}\right) - x_{23}\left(a_{32}d_{44} - x_{34}x_{42}\right) + x_{24}\left(a_{32}x_{43} - d_{33}x_{42}\right)$$

$$\left|A(2,3,5)\right| = d_{22}\left(d_{33}d_{55} - x_{35}a_{53}\right) - x_{23}\left(a_{32}d_{55} - x_{35}x_{52}\right) + x_{25}\left(a_{32}a_{53} - d_{33}x_{52}\right)$$

$$\left|A(2,4,5)\right| = d_{22}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{24}\left(x_{42}d_{55} - a_{45}x_{52}\right) + x_{25}\left(x_{42}x_{54} - d_{44}x_{52}\right)$$

$$\left|A(3,4,5)\right| = d_{33}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{34}\left(x_{43}d_{55} - a_{45}a_{53}\right) + x_{35}\left(x_{43}x_{54} - d_{44}a_{53}\right)$$

$$\begin{aligned}\left|A(1,2,3,4)\right| = {} & \left[d_{11}\left\{d_{22}\left(d_{33}d_{44} - x_{34}x_{43}\right) - x_{23}\left(a_{32}d_{44} - x_{34}x_{42}\right) + x_{24}\left(a_{32}x_{43} - d_{33}x_{42}\right)\right\}\right] \\ & -\left[x_{12}\left\{x_{21}\left(d_{33}d_{44} - x_{34}x_{43}\right) - x_{23}\left(x_{31}d_{44} - x_{34}x_{41}\right) + x_{24}\left(x_{31}x_{43} - d_{33}x_{41}\right)\right\}\right] \\ & +\left[x_{13}\left\{x_{21}\left(a_{32}d_{44} - x_{34}x_{42}\right) - d_{22}\left(x_{31}d_{44} - x_{34}x_{41}\right) + x_{24}\left(x_{31}x_{42} - a_{32}x_{41}\right)\right\}\right] \\ & -\left[x_{14}\left\{x_{21}\left(a_{32}x_{43} - d_{33}x_{42}\right) - d_{22}\left(x_{31}x_{43} - d_{33}x_{41}\right) + x_{23}\left(x_{31}x_{42} - a_{32}x_{41}\right)\right\}\right]\end{aligned}$$

$$\begin{aligned}\left|A(1,2,3,5)\right| = {} & \left[d_{11}\left\{d_{22}\left(d_{33}d_{55} - x_{35}a_{53}\right) - x_{23}\left(a_{32}d_{55} - x_{35}x_{52}\right) + x_{25}\left(a_{32}a_{53} - d_{33}x_{52}\right)\right\}\right] \\ & -\left[x_{12}\left\{x_{21}\left(d_{33}d_{55} - x_{35}a_{53}\right) - x_{23}\left(x_{31}d_{55} - x_{35}a_{51}\right) + x_{25}\left(x_{31}a_{53} - d_{33}a_{51}\right)\right\}\right] \\ & +\left[x_{13}\left\{x_{21}\left(a_{32}d_{55} - x_{35}x_{52}\right) - d_{22}\left(x_{31}d_{55} - x_{35}a_{51}\right) + x_{25}\left(x_{31}x_{52} - a_{32}a_{51}\right)\right\}\right] \\ & -\left[x_{15}\left\{x_{21}\left(a_{32}a_{53} - d_{33}x_{52}\right) - d_{22}\left(x_{31}a_{53} - d_{33}a_{51}\right) + x_{23}\left(x_{31}x_{52} - a_{32}a_{51}\right)\right\}\right]\end{aligned}$$

$$\begin{aligned}\left|A(1,2,4,5)\right| = {} & \left[d_{11}\left\{d_{22}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{24}\left(x_{42}d_{55} - a_{45}x_{52}\right) + x_{25}\left(x_{42}x_{54} - d_{44}x_{52}\right)\right\}\right] \\ & -\left[x_{12}\left\{x_{21}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{24}\left(x_{41}d_{55} - a_{45}a_{51}\right) + x_{25}\left(x_{41}x_{54} - d_{44}a_{51}\right)\right\}\right] \\ & +\left[x_{14}\left\{x_{21}\left(x_{42}d_{55} - a_{45}x_{52}\right) - d_{22}\left(x_{41}d_{55} - a_{45}a_{51}\right) + x_{25}\left(x_{41}x_{52} - x_{42}a_{51}\right)\right\}\right] \\ & -\left[x_{15}\left\{x_{21}\left(x_{42}x_{54} - d_{44}x_{52}\right) - d_{22}\left(x_{41}x_{54} - d_{44}a_{51}\right) + x_{24}\left(x_{41}x_{52} - x_{42}a_{51}\right)\right\}\right]\end{aligned}$$

$$\begin{aligned}\left|A(1,3,4,5)\right| = {} & \left[d_{11}\left\{d_{33}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{34}\left(x_{43}d_{55} - a_{45}a_{53}\right) + x_{35}\left(x_{43}x_{54} - d_{44}a_{53}\right)\right\}\right] \\ & -\left[x_{13}\left\{x_{31}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{34}\left(x_{41}d_{55} - a_{45}a_{51}\right) + x_{35}\left(x_{41}x_{54} - d_{44}a_{51}\right)\right\}\right] \\ & +\left[x_{14}\left\{x_{31}\left(x_{43}d_{55} - a_{45}a_{53}\right) - d_{33}\left(x_{41}d_{55} - a_{45}a_{51}\right) + x_{35}\left(x_{41}a_{53} - x_{43}a_{51}\right)\right\}\right] \\ & -\left[x_{15}\left\{x_{31}\left(x_{43}x_{54} - d_{44}a_{53}\right) - d_{33}\left(x_{41}x_{54} - d_{44}a_{51}\right) + x_{34}\left(x_{41}a_{53} - x_{43}a_{51}\right)\right\}\right]\end{aligned}$$

$$\begin{aligned}\left|A(2,3,4,5)\right| = {} & \left[d_{22}\left\{d_{33}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{34}\left(x_{43}d_{55} - a_{45}a_{53}\right) + x_{35}\left(x_{43}x_{54} - d_{44}a_{53}\right)\right\}\right] \\ & -\left[x_{23}\left\{a_{32}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{34}\left(x_{42}d_{55} - a_{45}x_{52}\right) + x_{35}\left(x_{42}x_{54} - d_{44}x_{52}\right)\right\}\right] \\ & +\left[x_{24}\left\{a_{32}\left(x_{43}d_{55} - a_{45}a_{53}\right) - d_{33}\left(x_{42}d_{55} - a_{45}x_{52}\right) + x_{35}\left(x_{42}a_{53} - x_{43}x_{52}\right)\right\}\right] \\ & -\left[x_{25}\left\{a_{32}\left(x_{43}x_{54} - d_{44}a_{53}\right) - d_{33}\left(x_{42}x_{54} - d_{44}x_{52}\right) + x_{34}\left(x_{42}a_{53} - x_{43}x_{52}\right)\right\}\right]\end{aligned}$$

$$\left|A(1,2,3,4,5)\right|$$

Since $x_{ij} = 0$

$$|A(1,2)| = d_{11}d_{22} \geq 0$$
$$|A(1,3)| = d_{11}d_{33} \geq 0$$
$$|A(1,4)| = d_{11}d_{44} \geq 0$$
$$|A(1,5)| = d_{11}d_{55} \geq 0$$
$$|A(2,3)| = d_{22}d_{33} \geq 0$$
$$|A(2,4)| = d_{22}d_{44} \geq 0$$
$$|A(2,5)| = d_{22}d_{55} \geq 0$$
$$|A(3,4)| = d_{33}d_{44} \geq 0$$
$$|A(3,5)| = d_{33}d_{55} \geq 0$$
$$|A(4,5)| = d_{44}d_{55} \geq 0$$
$$|A(1,2,3)| = d_{11}d_{22}d_{33} \geq 0$$
$$|A(1,2,4)| = d_{11}d_{22}d_{44} \geq 0$$
$$|A(1,2,5)| = d_{11}d_{22}d_{55} \geq 0$$
$$|A(1,3,4)| = d_{11}d_{33}d_{44} \geq 0$$
$$|A(1,3,5)| = d_{11}d_{33}d_{55} \geq 0$$
$$|A(1,4,5)| = d_{11}d_{44}d_{55} \geq 0$$
$$|A(2,3,4)| = d_{22}d_{33}d_{44} \geq 0$$
$$|A(2,3,5)| = d_{22}d_{33}d_{55} \geq 0$$
$$|A(2,4,5)| = d_{22}d_{44}d_{55} \geq 0$$
$$|A(3,4,5)| = d_{33}d_{44}d_{55} \geq 0$$
$$|A(1,2,3,4)| = d_{11}d_{22}d_{33}d_{44} \geq 0$$
$$|A(1,2,3,5)| = d_{11}d_{22}d_{33}d_{55} \geq 0$$
$$|A(1,2,4,5)| = d_{11}d_{22}d_{44}d_{55} \geq 0$$
$$|A(1,3,4,5)| = d_{11}d_{33}d_{44}d_{55} \geq 0$$
$$|A(2,3,4,5)| = d_{22}d_{33}d_{44}d_{55} \geq 0$$
$$|A(1,2,3,4,5)| = d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$$

The diagonal entries are specified and as stated earlier, are assumed to be all non-negative. Therefore, there is a zero completion into an M_0 -matrix for this case.

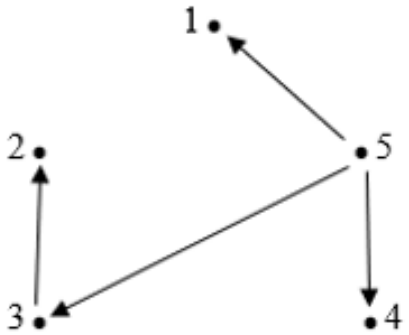

Fig 2:

$$B=\begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & a_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ a_{51} & x_{52} & a_{53} & a_{54} & d_{55} \end{bmatrix}$$

All entries in this partial matrix are the same as those in Digraph 2.1 except for a_{45} which is now x_{45} and x_{54} now a_{54} .Therefore, we only generate the principal minors of the sub-matrices where the numerals 4 and 5 both appear.

$$|B(4,5)| = d_{44}d_{55} - x_{45}a_{54}$$

$$|B(1,4,5)| = d_{11}(d_{44}d_{55} - x_{45}a_{54}) - x_{14}(x_{41}d_{55} - x_{45}a_{51}) + x_{15}(x_{41}a_{54} - d_{44}a_{51})$$

$$|B(2,4,5)| = d_{22}(d_{44}d_{55} - x_{45}a_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}a_{54} - d_{44}x_{52})$$

$$|B(3,4,5)| = d_{33}(d_{44}d_{55} - x_{45}a_{54}) - x_{34}(x_{43}d_{55} - x_{45}a_{53}) + x_{35}(x_{43}a_{54} - d_{44}a_{53})$$

$$\begin{aligned} |B(1,2,4,5)| = {} & \left[d_{11}\{d_{22}(d_{44}d_{55} - x_{45}a_{54}) - x_{24}(x_{42}d_{55} - x_{45}x_{52}) + x_{25}(x_{42}a_{54} - d_{44}x_{52})\}\right] \\ & -\left[x_{12}\{x_{21}(d_{44}d_{55} - x_{45}a_{54}) - x_{24}(x_{41}d_{55} - x_{45}a_{51}) + x_{25}(x_{41}a_{54} - d_{44}a_{51})\}\right] \\ & +\left[x_{14}\{x_{21}(x_{42}d_{55} - x_{45}x_{52}) - d_{22}(x_{41}d_{55} - x_{45}a_{51}) + x_{25}(x_{41}x_{52} - x_{42}a_{51})\}\right] \\ & -\left[x_{15}\{x_{21}(x_{42}a_{54} - d_{44}x_{52}) - d_{22}(x_{41}a_{54} - d_{44}a_{51}) + x_{24}(x_{41}x_{52} - x_{42}a_{51})\}\right] \end{aligned}$$

$$\begin{aligned} |B(1,3,4,5)| = {} & \left[d_{11}\{d_{33}(d_{44}d_{55} - x_{45}a_{54}) - x_{34}(x_{43}d_{55} - x_{45}a_{53}) + x_{35}(x_{43}a_{54} - d_{44}a_{53})\}\right] \\ & -\left[x_{13}\{x_{31}(d_{44}d_{55} - x_{45}a_{54}) - x_{34}(x_{41}d_{55} - x_{45}a_{51}) + x_{35}(x_{41}a_{54} - d_{44}a_{51})\}\right] \\ & +\left[x_{14}\{x_{31}(x_{43}d_{55} - x_{45}a_{53}) - d_{33}(x_{41}d_{55} - x_{45}a_{51}) + x_{35}(x_{41}a_{53} - x_{43}a_{51})\}\right] \\ & -\left[x_{15}\{x_{31}(x_{43}a_{54} - d_{44}a_{53}) - d_{33}(x_{41}a_{54} - d_{44}a_{51}) + x_{34}(x_{41}a_{53} - x_{43}a_{51})\}\right] \end{aligned}$$

$$\begin{aligned} |B(2,3,4,5)| = {} & \left[d_{22}\{d_{33}(d_{44}d_{55} - x_{45}a_{54}) - x_{34}(x_{43}d_{55} - x_{45}a_{53}) + x_{35}(x_{43}a_{54} - d_{44}a_{53})\}\right] \\ & -\left[x_{23}\{a_{32}(d_{44}d_{55} - x_{45}a_{54}) - x_{34}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}a_{54} - d_{44}x_{52})\}\right] \\ & +\left[x_{24}\{a_{32}(x_{43}d_{55} - x_{45}a_{53}) - d_{33}(x_{42}d_{55} - x_{45}x_{52}) + x_{35}(x_{42}a_{53} - x_{43}x_{52})\}\right] \\ & -\left[x_{25}\{a_{32}(x_{43}a_{54} - d_{44}a_{53}) - d_{33}(x_{42}a_{54} - d_{44}x_{52}) + x_{34}(x_{42}a_{53} - x_{43}x_{52})\}\right] \end{aligned}$$

$$
\begin{aligned}
\left|B(1,2,3,4,5)\right| = {} & d_{11}\begin{bmatrix} d_{22}\{d_{33}(d_{44}d_{55}-x_{45}a_{54})-x_{34}(x_{43}d_{55}-x_{45}a_{53})+x_{35}(x_{43}a_{54}-d_{44}a_{53})\} \\ -x_{23}\{a_{32}(d_{44}d_{55}-x_{45}a_{54})-x_{34}(x_{42}d_{55}-x_{45}x_{52})+x_{35}(x_{42}a_{54}-d_{44}x_{52})\} \\ +x_{24}\{a_{32}(x_{43}d_{55}-x_{45}a_{53})-d_{33}(x_{42}d_{55}-x_{45}x_{52})+x_{35}(x_{42}a_{53}-x_{43}x_{52})\} \\ -x_{25}\{a_{32}(x_{43}a_{54}-d_{44}a_{53})-d_{33}(x_{42}a_{54}-d_{44}x_{52})+x_{34}(x_{42}a_{53}-x_{43}x_{52})\} \end{bmatrix} \\
& -x_{12}\begin{bmatrix} x_{21}\{d_{33}(d_{44}d_{55}-x_{45}a_{54})-x_{34}(x_{43}d_{55}-x_{45}a_{53})+x_{35}(x_{43}a_{54}-d_{44}a_{53})\} \\ -x_{23}\{x_{31}(d_{44}d_{55}-x_{45}a_{54})-x_{34}(x_{41}d_{55}-x_{45}a_{51})+x_{35}(x_{41}a_{54}-d_{44}a_{51})\} \\ +x_{24}\{x_{31}(x_{43}d_{55}-x_{45}a_{53})-d_{33}(x_{41}d_{55}-x_{45}a_{51})+x_{35}(x_{41}a_{53}-x_{43}a_{51})\} \\ -x_{25}\{x_{31}(x_{43}a_{54}-d_{44}a_{53})-d_{33}(x_{41}a_{54}-d_{44}a_{51})+x_{34}(x_{41}a_{53}-x_{43}a_{51})\} \end{bmatrix} \\
& +x_{13}\begin{bmatrix} x_{21}\{a_{32}(d_{44}d_{55}-x_{45}a_{54})-x_{34}(x_{42}d_{55}-x_{45}x_{52})+x_{35}(x_{42}a_{54}-d_{44}x_{52})\} \\ -d_{22}\{x_{31}(d_{44}d_{55}-x_{45}a_{54})-x_{34}(x_{41}d_{55}-x_{45}a_{51})+x_{35}(x_{41}a_{54}-d_{44}a_{51})\} \\ +x_{24}\{x_{31}(x_{42}d_{55}-x_{45}x_{52})-a_{32}(x_{41}d_{55}-x_{45}a_{51})+x_{35}(x_{41}x_{52}-x_{42}a_{51})\} \\ -x_{25}\{x_{31}(x_{42}a_{54}-d_{44}x_{52})-a_{32}(x_{41}a_{54}-d_{44}a_{51})+x_{34}(x_{41}x_{52}-x_{42}a_{51})\} \end{bmatrix} \\
& -x_{14}\begin{bmatrix} x_{21}\{a_{32}(x_{43}d_{55}-x_{45}a_{53})-d_{33}(x_{42}d_{55}-x_{45}x_{52})+x_{35}(x_{42}a_{53}-x_{43}x_{52})\} \\ -d_{22}\{x_{31}(x_{43}d_{55}-x_{45}a_{53})-d_{33}(x_{41}d_{55}-x_{45}a_{51})+x_{35}(x_{41}a_{53}-x_{43}a_{51})\} \\ +x_{23}\{x_{31}(x_{42}d_{55}-x_{45}x_{52})-a_{32}(x_{41}d_{55}-x_{45}a_{51})+x_{35}(x_{41}x_{52}-x_{42}a_{51})\} \\ -x_{25}\{x_{31}(x_{42}a_{53}-x_{43}x_{52})-a_{32}(x_{41}a_{53}-x_{43}a_{51})+d_{33}(x_{41}x_{52}-x_{42}a_{51})\} \end{bmatrix} \\
& +x_{15}\begin{bmatrix} x_{21}\{a_{32}(x_{43}a_{54}-d_{44}a_{53})-d_{33}(x_{42}a_{54}-d_{44}x_{52})+x_{34}(x_{42}a_{53}-x_{43}x_{52})\} \\ -d_{22}\{x_{31}(x_{43}a_{54}-d_{44}a_{53})-d_{33}(x_{41}a_{54}-d_{44}a_{51})+x_{34}(x_{41}a_{53}-x_{43}a_{51})\} \\ +x_{23}\{x_{31}(x_{42}a_{54}-d_{44}x_{52})-a_{32}(x_{41}a_{54}-d_{44}a_{51})+x_{34}(x_{41}x_{52}-x_{42}a_{51})\} \\ -x_{24}\{x_{31}(x_{42}a_{53}-x_{43}x_{52})-a_{32}(x_{41}a_{53}-x_{43}a_{51})+d_{33}(x_{41}x_{52}-x_{42}a_{51})\} \end{bmatrix}
\end{aligned}
$$

Since $x_{ij}=0$

$$\left|B(4,5)\right| = d_{44}d_{55} \geq 0$$

$$\left|B(1,4,5)\right| = d_{11}d_{44}d_{55} \geq 0$$

$$\left|B(2,4,5)\right| = d_{22}d_{44}d_{55} \geq 0$$

$$\left|B(3,4,5)\right| = d_{33}d_{44}d_{55} \geq 0$$

$$\left|B(1,2,4,5)\right| = d_{11}d_{22}d_{44}d_{55} \geq 0$$

$$\left|B(1,3,4,5)\right| = d_{11}d_{33}d_{44}d_{55} \geq 0$$

$$\left|B(2,3,4,5)\right| = d_{22}d_{33}d_{44}d_{55} \geq 0$$

$$\left|B(1,2,3,4,5)\right| = d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$$

The principal minors are all non- negative. Thus, zero completion into an M_0 -matrix.

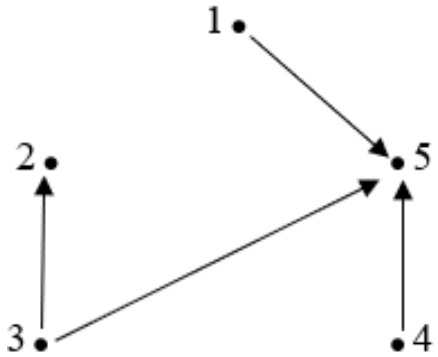

Fig 3:

$$C=\begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & a_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & a_{32} & d_{33} & x_{34} & a_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

All entries in this matrix are the same as those in our first case except for x_{15} which is now a_{15}, x_{35} now a_{35}, a_{51} now x_{51} and a_{53} now x_{53}.We therefore only consider sub-matrices where the numerals 1 and 5 or 3 and 5 are mentioned.

$$\left|C(1,5)\right| = d_{11}d_{55} - a_{15}x_{51}$$

$$\left|C(3,5)\right| = d_{33}d_{55} - a_{35}x_{53}$$

$$\left|C(1,2,5)\right| = d_{11}\left(d_{22}d_{55} - x_{25}x_{52}\right) - x_{12}\left(x_{21}d_{55} - x_{25}x_{51}\right) + a_{15}\left(x_{21}x_{52} - d_{22}x_{51}\right)$$

$$\left|C(1,3,5)\right| = d_{11}\left(d_{33}d_{55} - a_{35}x_{53}\right) - x_{13}\left(x_{31}d_{55} - a_{35}x_{51}\right) + a_{15}\left(x_{31}x_{53} - d_{33}x_{51}\right)$$

$$\left|C(2,3,5)\right| = d_{22}\left(d_{33}d_{55} - a_{35}x_{53}\right) - x_{23}\left(a_{32}d_{55} - a_{35}x_{52}\right) + x_{25}\left(a_{32}x_{53} - d_{33}x_{52}\right)$$

$$\left|C(3,4,5)\right| = d_{33}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{34}\left(x_{43}d_{55} - a_{45}x_{53}\right) + a_{35}\left(x_{43}x_{54} - d_{44}x_{53}\right)$$

$$\begin{aligned} \left|C(1,2,3,5)\right| = {} & \left[d_{11}\left\{d_{22}\left(d_{33}d_{55} - a_{35}x_{53}\right) - x_{23}\left(a_{32}d_{55} - a_{35}x_{52}\right) + x_{25}\left(a_{32}x_{53} - d_{33}x_{52}\right)\right\}\right] \\ & -\left[x_{12}\left\{x_{21}\left(d_{33}d_{55} - a_{35}x_{53}\right) - x_{23}\left(x_{31}d_{55} - a_{35}x_{51}\right) + x_{25}\left(x_{31}x_{53} - d_{33}x_{51}\right)\right\}\right] \\ & +\left[x_{13}\left\{x_{21}\left(a_{32}d_{55} - a_{35}x_{52}\right) - d_{22}\left(x_{31}d_{55} - a_{35}x_{51}\right) + x_{25}\left(x_{31}x_{52} - a_{32}x_{51}\right)\right\}\right] \\ & -\left[a_{15}\left\{x_{21}\left(a_{32}x_{53} - d_{33}x_{52}\right) - d_{22}\left(x_{31}x_{53} - d_{33}a_{51}\right) + x_{23}\left(x_{31}x_{52} - a_{32}x_{51}\right)\right\}\right] \end{aligned}$$

$$\begin{aligned} \left|C(1,2,4,5)\right| = {} & \left[d_{11}\left\{d_{22}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{24}\left(x_{42}d_{55} - a_{45}x_{52}\right) + x_{25}\left(x_{42}x_{54} - d_{44}x_{52}\right)\right\}\right] \\ & -\left[x_{12}\left\{x_{21}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{24}\left(x_{41}d_{55} - a_{45}x_{51}\right) + x_{25}\left(x_{41}x_{54} - d_{44}x_{51}\right)\right\}\right] \\ & +\left[x_{14}\left\{x_{21}\left(x_{42}d_{55} - a_{45}x_{52}\right) - d_{22}\left(x_{41}d_{55} - a_{45}x_{51}\right) + x_{25}\left(x_{41}x_{52} - x_{42}x_{51}\right)\right\}\right] \\ & -\left[a_{15}\left\{x_{21}\left(x_{42}x_{54} - d_{44}x_{52}\right) - d_{22}\left(x_{41}x_{54} - d_{44}x_{51}\right) + x_{24}\left(x_{41}x_{52} - x_{42}x_{51}\right)\right\}\right] \end{aligned}$$

$$\left|C(1,3,4,5)\right| = \begin{matrix} \left[d_{11}\left\{d_{33}\left(d_{44}d_{55}-a_{45}x_{54}\right)-x_{34}\left(x_{43}d_{55}-a_{45}x_{53}\right)+a_{35}\left(x_{43}x_{54}-d_{44}x_{53}\right)\right\}\right] \\ -\left[x_{13}\left\{x_{31}\left(d_{44}d_{55}-a_{45}x_{54}\right)-x_{34}\left(x_{41}d_{55}-a_{45}x_{51}\right)+a_{35}\left(x_{41}x_{54}-d_{44}x_{51}\right)\right\}\right] \\ +\left[x_{14}\left\{x_{31}\left(x_{43}d_{55}-a_{45}x_{53}\right)-d_{33}\left(x_{41}d_{55}-a_{45}x_{51}\right)+a_{35}\left(x_{41}x_{53}-x_{43}x_{51}\right)\right\}\right] \\ -\left[a_{15}\left\{x_{31}\left(x_{43}x_{54}-d_{44}x_{53}\right)-d_{33}\left(x_{41}x_{54}-d_{44}x_{51}\right)+x_{34}\left(x_{41}x_{53}-x_{43}x_{51}\right)\right\}\right] \end{matrix}$$

$$\left|C(2,3,4,5)\right| = \begin{matrix} \left[d_{22}\left\{d_{33}\left(d_{44}d_{55}-a_{45}x_{54}\right)-x_{34}\left(x_{43}d_{55}-a_{45}x_{53}\right)+a_{35}\left(x_{43}x_{54}-d_{44}x_{53}\right)\right\}\right] \\ -\left[x_{23}\left\{a_{32}\left(d_{44}d_{55}-a_{45}x_{54}\right)-x_{34}\left(x_{42}d_{55}-a_{45}x_{52}\right)+a_{35}\left(x_{42}x_{54}-d_{44}x_{52}\right)\right\}\right] \\ +\left[x_{24}\left\{a_{32}\left(x_{43}d_{55}-a_{45}x_{53}\right)-d_{33}\left(x_{42}d_{55}-a_{45}x_{52}\right)+a_{35}\left(x_{42}x_{53}-x_{43}x_{52}\right)\right\}\right] \\ -\left[x_{25}\left\{a_{32}\left(x_{43}x_{54}-d_{44}x_{53}\right)-d_{33}\left(x_{42}x_{54}-d_{44}x_{52}\right)+x_{34}\left(x_{42}x_{53}-x_{43}x_{52}\right)\right\}\right] \end{matrix}$$

Since $x_{ij} = 0$

$$\left|C(1,5)\right| = d_{11}d_{55} \geq 0$$

$$\left|C(3,5)\right| = d_{33}d_{55} \geq 0$$

$$\left|C(1,2,5)\right| = d_{11}d_{22}d_{55} \geq 0$$

$$\left|C(1,3,5)\right| = d_{11}d_{33}d_{55} \geq 0$$

$$\left|C(1,4,5)\right| = d_{11}d_{44}d_{55} \geq 0$$

$$\left|C(2,3,5)\right| = d_{22}d_{33}d_{55} \geq 0$$

$$\left|C(3,4,5)\right| = d_{33}d_{44}d_{55} \geq 0$$

$$\left|C(1,2,3,5)\right| = d_{11}d_{22}d_{33}d_{55} \geq 0$$

$$\left|C(1,2,4,5)\right| = d_{11}d_{22}d_{44}d_{55} \geq 0$$

$$
\begin{aligned}
|C(1,2,3,4,5)| = {} & d_{11}\begin{bmatrix} d_{22}\{d_{33}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{43}d_{55}-a_{45}x_{53})+a_{35}(x_{43}x_{54}-d_{44}x_{53})\} \\ -x_{23}\{a_{32}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{42}d_{55}-a_{45}x_{52})+a_{35}(x_{42}x_{54}-d_{44}x_{52})\} \\ +x_{24}\{a_{32}(x_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{42}d_{55}-a_{45}x_{52})+a_{35}(x_{42}x_{53}-x_{43}x_{52})\} \\ -x_{25}\{a_{32}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{42}x_{54}-d_{44}x_{52})+x_{34}(x_{42}x_{53}-x_{43}x_{52})\} \end{bmatrix} \\
& -x_{12}\begin{bmatrix} x_{21}\{d_{33}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{43}d_{55}-a_{45}x_{53})+a_{35}(x_{43}x_{54}-d_{44}x_{53})\} \\ -x_{23}\{x_{31}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{41}d_{55}-a_{45}x_{51})+a_{35}(x_{41}x_{54}-d_{44}x_{51})\} \\ +x_{24}\{x_{31}(x_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{41}d_{55}-a_{45}x_{51})+a_{35}(x_{41}x_{53}-x_{43}x_{51})\} \\ -x_{25}\{x_{31}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{41}x_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{53}-x_{43}x_{51})\} \end{bmatrix} \\
& +x_{13}\begin{bmatrix} x_{21}\{a_{32}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{42}d_{55}-a_{45}x_{52})+a_{35}(x_{42}x_{54}-d_{44}x_{52})\} \\ -d_{22}\{x_{31}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{41}d_{55}-a_{45}x_{51})+a_{35}(x_{41}x_{54}-d_{44}x_{51})\} \\ +x_{24}\{x_{31}(x_{42}d_{55}-a_{45}x_{52})-a_{32}(x_{41}d_{55}-a_{45}x_{51})+a_{35}(x_{41}x_{52}-x_{42}x_{51})\} \\ -x_{25}\{x_{31}(x_{42}x_{54}-d_{44}x_{52})-a_{32}(x_{41}x_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{52}-x_{42}x_{51})\} \end{bmatrix} \\
& -x_{14}\begin{bmatrix} x_{21}\{a_{32}(x_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{42}d_{55}-a_{45}x_{52})+a_{35}(x_{42}x_{53}-x_{43}x_{52})\} \\ -d_{22}\{x_{31}(x_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{41}d_{55}-a_{45}x_{51})+a_{35}(x_{41}x_{53}-x_{43}x_{51})\} \\ +x_{23}\{x_{31}(x_{42}d_{55}-a_{45}x_{52})-a_{32}(x_{41}d_{55}-a_{45}x_{51})+a_{35}(x_{41}x_{52}-x_{42}x_{51})\} \\ -x_{25}\{x_{31}(x_{42}x_{53}-x_{43}x_{52})-a_{32}(x_{41}x_{53}-x_{43}x_{51})+d_{33}(x_{41}x_{52}-x_{42}x_{51})\} \end{bmatrix} \\
& +a_{15}\begin{bmatrix} x_{21}\{a_{32}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{42}x_{54}-d_{44}x_{52})+x_{34}(x_{42}x_{53}-x_{43}x_{52})\} \\ -d_{22}\{x_{31}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{41}x_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{53}-x_{43}x_{51})\} \\ +x_{23}\{x_{31}(x_{42}x_{54}-d_{44}x_{52})-a_{32}(x_{41}x_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{52}-x_{42}x_{51})\} \\ -x_{24}\{x_{31}(x_{42}x_{53}-x_{43}x_{52})-a_{32}(x_{41}x_{53}-x_{43}x_{51})+d_{33}(x_{41}x_{52}-x_{42}x_{51})\} \end{bmatrix}
\end{aligned}
$$

$$|C(1,3,4,5)| = d_{11}d_{33}d_{44}d_{55} \geq 0$$
$$|C(2,3,4,5)| = d_{22}d_{33}d_{44}d_{55} \geq 0$$
$$|C(1,2,3,4,5)| = d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$$

Hence, zero completion into an M_0 -matrix.

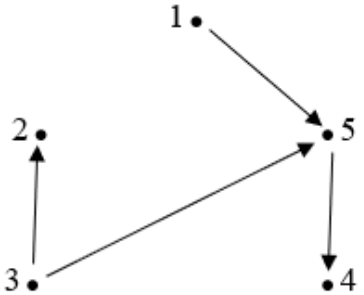

Fig 4:

$$D=\begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & a_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & a_{32} & d_{33} & x_{34} & a_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & a_{54} & d_{55} \end{bmatrix}$$

The entries in this matrix are the same as those in above case (2.3) except for a_{45} which is now x_{45} and x_{54} now a_{54}. We only consider sub-matrices with both numerals 4 and 5

$$\left|D(4,5)\right| = d_{44}d_{55} - x_{45}a_{54}$$

$$\left|D(1,4,5)\right| = d_{11}\left(d_{44}d_{55} - x_{45}a_{54}\right) - x_{14}\left(x_{41}d_{55} - x_{45}x_{51}\right) + a_{15}\left(x_{41}a_{54} - d_{44}x_{51}\right)$$

$$\left|D(2,4,5)\right| = d_{22}\left(d_{44}d_{55} - x_{45}a_{54}\right) - x_{24}\left(x_{42}d_{55} - x_{45}x_{52}\right) + x_{25}\left(x_{42}a_{54} - d_{44}x_{52}\right)$$

$$\left|D(3,4,5)\right| = d_{33}\left(d_{44}d_{55} - x_{45}a_{54}\right) - x_{34}\left(x_{43}d_{55} - x_{45}x_{53}\right) + a_{35}\left(x_{43}a_{54} - d_{44}x_{53}\right)$$

$$\left|D(1,2,4,5)\right| = \begin{aligned} &\left[d_{11}\left\{d_{22}\left(d_{44}d_{55} - x_{45}a_{54}\right) - x_{24}\left(x_{42}d_{55} - x_{45}x_{52}\right) + x_{25}\left(x_{42}a_{54} - d_{44}x_{52}\right)\right\}\right] \\ &-\left[x_{12}\left\{x_{21}\left(d_{44}d_{55} - x_{45}a_{54}\right) - x_{24}\left(x_{41}d_{55} - x_{45}x_{51}\right) + x_{25}\left(x_{41}a_{54} - d_{44}x_{51}\right)\right\}\right] \\ &+\left[x_{14}\left\{x_{21}\left(x_{42}d_{55} - x_{45}x_{52}\right) - d_{22}\left(x_{41}d_{55} - x_{45}x_{51}\right) + x_{25}\left(x_{41}x_{52} - x_{42}x_{51}\right)\right\}\right] \\ &-\left[a_{15}\left\{x_{21}\left(x_{42}a_{54} - d_{44}x_{52}\right) - d_{22}\left(x_{41}a_{54} - d_{44}x_{51}\right) + x_{24}\left(x_{41}x_{52} - x_{42}x_{51}\right)\right\}\right] \end{aligned}$$

$$\left|D(1,3,4,5)\right| = \begin{aligned} &\left[d_{11}\left\{d_{33}\left(d_{44}d_{55} - x_{45}a_{54}\right) - x_{34}\left(x_{43}d_{55} - x_{45}x_{53}\right) + a_{35}\left(x_{43}a_{54} - d_{44}x_{53}\right)\right\}\right] \\ &-\left[x_{13}\left\{x_{31}\left(d_{44}d_{55} - x_{45}a_{54}\right) - x_{34}\left(x_{41}d_{55} - x_{45}x_{51}\right) + a_{35}\left(x_{41}a_{54} - d_{44}x_{51}\right)\right\}\right] \\ &+\left[x_{14}\left\{x_{31}\left(x_{43}d_{55} - x_{45}x_{53}\right) - d_{33}\left(x_{41}d_{55} - x_{45}x_{51}\right) + a_{35}\left(x_{41}x_{53} - x_{43}x_{51}\right)\right\}\right] \\ &-\left[a_{15}\left\{x_{31}\left(x_{43}a_{54} - d_{44}x_{53}\right) - d_{33}\left(x_{41}a_{54} - d_{44}x_{51}\right) + x_{34}\left(x_{41}x_{53} - x_{43}x_{51}\right)\right\}\right] \end{aligned}$$

$$\left|D(2,3,4,5)\right| = \begin{aligned} &\left[d_{22}\left\{d_{33}\left(d_{44}d_{55} - x_{45}a_{54}\right) - x_{34}\left(x_{43}d_{55} - x_{45}x_{53}\right) + a_{35}\left(x_{43}a_{54} - d_{44}x_{53}\right)\right\}\right] \\ &-\left[x_{23}\left\{a_{32}\left(d_{44}d_{55} - x_{45}a_{54}\right) - x_{34}\left(x_{42}d_{55} - x_{45}x_{52}\right) + a_{35}\left(x_{42}a_{54} - d_{44}x_{52}\right)\right\}\right] \\ &+\left[x_{24}\left\{a_{32}\left(x_{43}d_{55} - x_{45}x_{53}\right) - d_{33}\left(x_{42}d_{55} - x_{45}x_{52}\right) + a_{35}\left(x_{42}x_{53} - x_{43}x_{52}\right)\right\}\right] \\ &-\left[x_{25}\left\{a_{32}\left(x_{43}a_{54} - d_{44}x_{53}\right) - d_{33}\left(x_{42}a_{54} - d_{44}x_{52}\right) + x_{34}\left(x_{42}x_{53} - x_{43}x_{52}\right)\right\}\right] \end{aligned}$$

$$
\begin{aligned}
\left|D(1,2,3,4,5)\right| = \;
& d_{11}\begin{bmatrix}
d_{22}\left\{d_{33}\left(d_{44}d_{55}-x_{45}a_{54}\right)-x_{34}\left(x_{43}d_{55}-x_{45}x_{53}\right)+a_{35}\left(x_{43}a_{54}-d_{44}x_{53}\right)\right\}\\
-x_{23}\left\{a_{32}\left(d_{44}d_{55}-x_{45}a_{54}\right)-x_{34}\left(x_{42}d_{55}-x_{45}x_{52}\right)+a_{35}\left(x_{42}a_{54}-d_{44}x_{52}\right)\right\}\\
+x_{24}\left\{a_{32}\left(x_{43}d_{55}-x_{45}x_{53}\right)-d_{33}\left(x_{42}d_{55}-x_{45}x_{52}\right)+a_{35}\left(x_{42}x_{53}-x_{43}x_{52}\right)\right\}\\
-x_{25}\left\{a_{32}\left(x_{43}a_{54}-d_{44}x_{53}\right)-d_{33}\left(x_{42}a_{54}-d_{44}x_{52}\right)+x_{34}\left(x_{42}x_{53}-x_{43}x_{52}\right)\right\}
\end{bmatrix}\\
& -x_{12}\begin{bmatrix}
x_{21}\left\{d_{33}\left(d_{44}d_{55}-x_{45}a_{54}\right)-x_{34}\left(x_{43}d_{55}-x_{45}x_{53}\right)+a_{35}\left(x_{43}a_{54}-d_{44}x_{53}\right)\right\}\\
-x_{23}\left\{x_{31}\left(d_{44}d_{55}-x_{45}a_{54}\right)-x_{34}\left(x_{41}d_{55}-x_{45}x_{51}\right)+a_{35}\left(x_{41}a_{54}-d_{44}x_{51}\right)\right\}\\
+x_{24}\left\{x_{31}\left(x_{43}d_{55}-x_{45}x_{53}\right)-d_{33}\left(x_{41}d_{55}-x_{45}x_{51}\right)+a_{35}\left(x_{41}x_{53}-x_{43}x_{51}\right)\right\}\\
-x_{25}\left\{x_{31}\left(x_{43}a_{54}-d_{44}x_{53}\right)-d_{33}\left(x_{41}a_{54}-d_{44}x_{51}\right)+x_{34}\left(x_{41}x_{53}-x_{43}x_{51}\right)\right\}
\end{bmatrix}\\
& +x_{13}\begin{bmatrix}
x_{21}\left\{a_{32}\left(d_{44}d_{55}-x_{45}a_{54}\right)-x_{34}\left(x_{42}d_{55}-x_{45}x_{52}\right)+a_{35}\left(x_{42}a_{54}-d_{44}x_{52}\right)\right\}\\
-d_{22}\left\{x_{31}\left(d_{44}d_{55}-x_{45}a_{54}\right)-x_{34}\left(x_{41}d_{55}-x_{45}x_{51}\right)+a_{35}\left(x_{41}a_{54}-d_{44}x_{51}\right)\right\}\\
+x_{24}\left\{x_{31}\left(x_{42}d_{55}-x_{45}x_{52}\right)-a_{32}\left(x_{41}d_{55}-x_{45}x_{51}\right)+a_{35}\left(x_{41}x_{52}-x_{42}x_{51}\right)\right\}\\
-x_{25}\left\{x_{31}\left(x_{42}a_{54}-d_{44}x_{52}\right)-a_{32}\left(x_{41}a_{54}-d_{44}x_{51}\right)+x_{34}\left(x_{41}x_{52}-x_{42}x_{51}\right)\right\}
\end{bmatrix}\\
& -x_{14}\begin{bmatrix}
x_{21}\left\{a_{32}\left(x_{43}d_{55}-x_{45}x_{53}\right)-d_{33}\left(x_{42}d_{55}-x_{45}x_{52}\right)+a_{35}\left(x_{42}x_{53}-x_{43}x_{52}\right)\right\}\\
-d_{22}\left\{x_{31}\left(x_{43}d_{55}-x_{45}x_{53}\right)-d_{33}\left(x_{41}d_{55}-x_{45}x_{51}\right)+a_{35}\left(x_{41}x_{53}-x_{43}x_{51}\right)\right\}\\
+x_{23}\left\{x_{31}\left(x_{42}d_{55}-x_{45}x_{52}\right)-a_{32}\left(x_{41}d_{55}-x_{45}x_{51}\right)+a_{35}\left(x_{41}x_{52}-x_{42}x_{51}\right)\right\}\\
-x_{25}\left\{x_{31}\left(x_{42}x_{53}-x_{43}x_{52}\right)-a_{32}\left(x_{41}x_{53}-x_{43}x_{51}\right)+d_{33}\left(x_{41}x_{52}-x_{42}x_{51}\right)\right\}
\end{bmatrix}\\
& +a_{15}\begin{bmatrix}
x_{21}\left\{a_{32}\left(x_{43}a_{54}-d_{44}x_{53}\right)-d_{33}\left(x_{42}a_{54}-d_{44}x_{52}\right)+x_{34}\left(x_{42}x_{53}-x_{43}x_{52}\right)\right\}\\
-d_{22}\left\{x_{31}\left(x_{43}a_{54}-d_{44}x_{53}\right)-d_{33}\left(x_{41}a_{54}-d_{44}x_{51}\right)+x_{34}\left(x_{41}x_{53}-x_{43}x_{51}\right)\right\}\\
+x_{23}\left\{x_{31}\left(x_{42}a_{54}-d_{44}x_{52}\right)-a_{32}\left(x_{41}a_{54}-d_{44}x_{51}\right)+x_{34}\left(x_{41}x_{52}-x_{42}x_{51}\right)\right\}\\
-x_{24}\left\{x_{31}\left(x_{42}x_{53}-x_{43}x_{52}\right)-a_{32}\left(x_{41}x_{53}-x_{43}x_{51}\right)+d_{33}\left(x_{41}x_{52}-x_{42}x_{51}\right)\right\}
\end{bmatrix}
\end{aligned}
$$

Since $x_{ij}=0$

$$\left|D(4,5)\right| = d_{44}d_{55} \geq 0$$

$$\left|D(1,4,5)\right| = d_{11}d_{44}d_{55} \geq 0$$

$$\left|D(2,4,5)\right| = d_{22}d_{44}d_{55} \geq 0$$

$$\left|D(3,4,5)\right| = d_{33}d_{44}d_{55} \geq 0$$

$$\left|D(1,2,4,5)\right| = d_{11}d_{22}d_{44}d_{55} \geq 0$$

$$\left|D(1,3,4,5)\right| = d_{11}d_{33}d_{44}d_{55} \geq 0$$

$$\left|D(2,3,4,5)\right| = d_{22}d_{33}d_{44}d_{55} \geq 0$$

$$\left|D(1,2,3,4,5)\right| = d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$$

All non-negative implying zero completion into an M_0 -matrix.

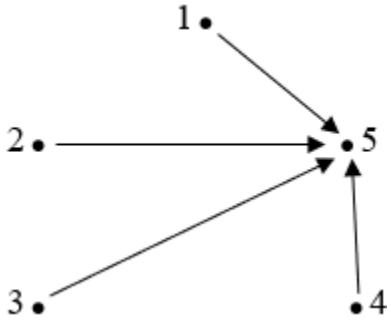

Fig 5:

$$E=\begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & a_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & a_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & a_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

All entries are the same as those in case 2.3 except for x_{25} which is now a_{25} and a_{32} now x_{32}.We consider only sub-matrices where the numerals 2 and 3 or 2 and 5.

$$|E(2,3)| = d_{22}d_{33} - x_{23}x_{32}$$

$$|E(2,5)| = d_{22}d_{55} - a_{25}x_{52}$$

$$|E(1,2,3)| = d_{11}(d_{22}d_{33} - x_{23}x_{32}) - x_{12}(x_{21}d_{33} - x_{23}x_{31}) + x_{13}(x_{21}x_{32} - d_{22}x_{31})$$

$$|E(1,2,5)| = d_{11}(d_{22}d_{55} - a_{25}x_{52}) - x_{12}(x_{21}d_{55} - a_{25}x_{51}) + a_{15}(x_{21}x_{52} - d_{22}x_{51})$$

$$|E(2,3,4)| = d_{22}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(x_{32}d_{44} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})$$

$$|E(2,3,5)| = d_{22}(d_{33}d_{55} - a_{35}x_{53}) - x_{23}(x_{32}d_{55} - a_{35}x_{52}) + a_{25}(x_{32}x_{53} - d_{33}x_{52})$$

$$|E(2,4,5)| = d_{22}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{42}d_{55} - a_{45}x_{52}) + a_{25}(x_{42}x_{54} - d_{44}x_{52})$$

$$|E(1,2,3,4)| = \begin{aligned} &\left[d_{11}\left\{d_{22}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(x_{32}d_{44} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})\right\}\right] \\ &-\left[x_{12}\left\{x_{21}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})\right\}\right] \\ &+\left[x_{13}\left\{x_{21}(x_{32}d_{44} - x_{34}x_{42}) - d_{22}(x_{31}d_{44} - x_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})\right\}\right] \\ &-\left[x_{14}\left\{x_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{31}x_{43} - d_{33}x_{41}) + x_{23}(x_{31}x_{42} - x_{32}x_{41})\right\}\right] \end{aligned}$$

$$|E(1,2,3,5)| = \begin{aligned} &\left[d_{11}\left\{d_{22}(d_{33}d_{55} - a_{35}x_{53}) - x_{23}(x_{32}d_{55} - a_{35}x_{52}) + a_{25}(x_{32}x_{53} - d_{33}x_{52})\right\}\right] \\ &-\left[x_{12}\left\{x_{21}(d_{33}d_{55} - a_{35}x_{53}) - x_{23}(x_{31}d_{55} - a_{35}x_{51}) + a_{25}(x_{31}x_{53} - d_{33}x_{51})\right\}\right] \\ &+\left[x_{13}\left\{x_{21}(x_{32}d_{55} - a_{35}x_{52}) - d_{22}(x_{31}d_{55} - a_{35}x_{51}) + a_{25}(x_{31}x_{52} - x_{32}x_{51})\right\}\right] \\ &-\left[a_{15}\left\{x_{21}(x_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}x_{51}) + x_{23}(x_{31}x_{52} - x_{32}x_{51})\right\}\right] \end{aligned}$$

$$|E(1,2,4,5)| = \begin{aligned} &\left[d_{11}\left\{d_{22}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{42}d_{55} - a_{45}x_{52}) + a_{25}(x_{42}x_{54} - d_{44}x_{52})\right\}\right] \\ &-\left[x_{12}\left\{x_{21}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{41}d_{55} - a_{45}x_{51}) + a_{25}(x_{41}x_{54} - d_{44}x_{51})\right\}\right] \\ &+\left[x_{14}\left\{x_{21}(x_{42}d_{55} - a_{45}x_{52}) - d_{22}(x_{41}d_{55} - a_{45}x_{51}) + a_{25}(x_{41}x_{52} - x_{42}x_{51})\right\}\right] \\ &-\left[a_{15}\left\{x_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}x_{51}) + x_{24}(x_{41}x_{52} - x_{42}x_{51})\right\}\right] \end{aligned}$$

$$\left|E(2,3,4,5)\right| = \begin{matrix} \left[d_{22}\left\{d_{33}\left(d_{44}d_{55}-a_{45}x_{54}\right)-x_{34}\left(x_{43}d_{55}-a_{45}x_{53}\right)+a_{35}\left(x_{43}x_{54}-d_{44}x_{53}\right)\right\}\right] \\ -\left[x_{23}\left\{x_{32}\left(d_{44}d_{55}-a_{45}x_{54}\right)-x_{34}\left(x_{42}d_{55}-a_{45}x_{52}\right)+a_{35}\left(x_{42}x_{54}-d_{44}x_{52}\right)\right\}\right] \\ +\left[x_{24}\left\{x_{32}\left(x_{43}d_{55}-a_{45}x_{53}\right)-d_{33}\left(x_{42}d_{55}-a_{45}x_{52}\right)+a_{35}\left(x_{42}x_{53}-x_{43}x_{52}\right)\right\}\right] \\ -\left[a_{25}\left\{x_{32}\left(x_{43}x_{54}-d_{44}x_{53}\right)-d_{33}\left(x_{42}x_{54}-d_{44}x_{52}\right)+x_{34}\left(x_{42}x_{53}-x_{43}x_{52}\right)\right\}\right] \end{matrix}$$

Since $x_{ij}=0$

$$\left|E(2,3)\right| = d_{22}d_{33} \geq 0$$

$$\left|E(2,5)\right| = d_{22}d_{55} \geq 0$$

$$\left|E(1,2,3)\right| = d_{11}d_{22}d_{33} \geq 0$$

$$\left|E(1,2,5)\right| = d_{11}d_{22}d_{55} \geq 0$$

$$\left|E(2,3,4)\right| = d_{22}d_{33}d_{44} \geq 0$$

$$\left|E(2,3,5)\right| = d_{22}d_{33}d_{55} \geq 0$$

$$\left|E(2,4,5)\right| = d_{22}d_{44}d_{55} \geq 0$$

$$\left|E(1,2,3,4)\right| = d_{11}d_{22}d_{33}d_{44} \geq 0$$

$$\left|E(1,2,3,5)\right| = d_{11}d_{22}d_{33}d_{55} \geq 0$$

$$\left|E(1,2,4,5)\right| = d_{11}d_{22}d_{44}d_{55} \geq 0$$

$$
\begin{aligned}
\left|E(1,2,3,4,5)\right| = {} & d_{11}\begin{bmatrix} d_{22}\{d_{33}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{43}d_{55}-a_{45}x_{53})+a_{35}(x_{43}x_{54}-d_{44}x_{53})\} \\ -x_{23}\{x_{32}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{42}d_{55}-a_{45}x_{52})+a_{35}(x_{42}x_{54}-d_{44}x_{52})\} \\ +x_{24}\{x_{32}(x_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{42}d_{55}-a_{45}x_{52})+a_{35}(x_{42}x_{53}-x_{43}x_{52})\} \\ -a_{25}\{x_{32}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{42}x_{54}-d_{44}x_{52})+x_{34}(x_{42}x_{53}-x_{43}x_{52})\} \end{bmatrix} \\
& -x_{12}\begin{bmatrix} x_{21}\{d_{33}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{43}d_{55}-a_{45}x_{53})+a_{35}(x_{43}x_{54}-d_{44}x_{53})\} \\ -x_{23}\{x_{31}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{41}d_{55}-a_{45}x_{51})+a_{35}(x_{41}x_{54}-d_{44}x_{51})\} \\ +x_{24}\{x_{31}(x_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{41}d_{55}-a_{45}x_{51})+a_{35}(x_{41}x_{53}-x_{43}x_{51})\} \\ -a_{25}\{x_{31}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{41}x_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{53}-x_{43}x_{51})\} \end{bmatrix} \\
& +x_{13}\begin{bmatrix} x_{21}\{x_{32}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{42}d_{55}-a_{45}x_{52})+a_{35}(x_{42}x_{54}-d_{44}x_{52})\} \\ -d_{22}\{x_{31}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{41}d_{55}-a_{45}x_{51})+a_{35}(x_{41}x_{54}-d_{44}x_{51})\} \\ +x_{24}\{x_{31}(x_{42}d_{55}-a_{45}x_{52})-x_{32}(x_{41}d_{55}-a_{45}x_{51})+a_{35}(x_{41}x_{52}-x_{42}x_{51})\} \\ -a_{25}\{x_{31}(x_{42}x_{54}-d_{44}x_{52})-x_{32}(x_{41}x_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{52}-x_{42}x_{51})\} \end{bmatrix} \\
& -x_{14}\begin{bmatrix} x_{21}\{x_{32}(x_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{42}d_{55}-a_{45}x_{52})+a_{35}(x_{42}x_{53}-x_{43}x_{52})\} \\ -d_{22}\{x_{31}(x_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{41}d_{55}-a_{45}x_{51})+a_{35}(x_{41}x_{53}-x_{43}x_{51})\} \\ +x_{23}\{x_{31}(x_{42}d_{55}-a_{45}x_{52})-x_{32}(x_{41}d_{55}-a_{45}x_{51})+a_{35}(x_{41}x_{52}-x_{42}x_{51})\} \\ -a_{25}\{x_{31}(x_{42}x_{53}-x_{43}x_{52})-x_{32}(x_{41}x_{53}-x_{43}x_{51})+d_{33}(x_{41}x_{52}-x_{42}x_{51})\} \end{bmatrix} \\
& +a_{15}\begin{bmatrix} x_{21}\{x_{32}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{42}x_{54}-d_{44}x_{52})+x_{34}(x_{42}x_{53}-x_{43}x_{52})\} \\ -d_{22}\{x_{31}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{41}x_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{53}-x_{43}x_{51})\} \\ +x_{23}\{x_{31}(x_{42}x_{54}-d_{44}x_{52})-x_{32}(x_{41}x_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{52}-x_{42}x_{51})\} \\ -x_{24}\{x_{31}(x_{42}x_{53}-x_{43}x_{52})-x_{32}(x_{41}x_{53}-x_{43}x_{51})+d_{33}(x_{41}x_{52}-x_{42}x_{51})\} \end{bmatrix}
\end{aligned}
$$

$$\left|E(2,3,4,5)\right| = d_{22}d_{33}d_{44}d_{55} \geq 0$$

$$\left|E(1,2,3,4,5)\right| = d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$$

The principal minors all non-negative, zero completion into an M_0 -matrix.

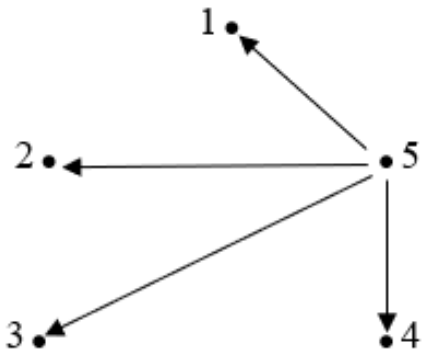

Fig 6:

Let $E^T = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & d_{55} \end{bmatrix}$ be a partial M_0-matrix representing the digraph above.

This matrix is a transpose of that in case 2.5 and since $\det E = \det E^T$,the digraph has M_0 - Completion.

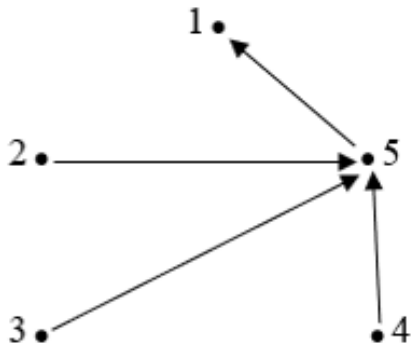

Fig 7:

Let $F = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & a_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & a_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ a_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$ be a partial M_0-matrix representing the digraph above.

In comparison to Digraph 2.5, the only changes are a_{15} to x_{15} and x_{51} to a_{51}. As such we only consider sub-matrices with 1 and 5

$$|F(1,5)| = d_{11}d_{55} - x_{15}a_{51}$$

$$|F(1,2,5)| = d_{11}(d_{22}d_{55} - a_{25}x_{52}) - x_{12}(x_{21}d_{55} - a_{25}a_{51}) + x_{15}(x_{21}x_{52} - d_{22}a_{51})$$

$$|F(1,3,5)| = d_{11}(d_{33}d_{55} - a_{35}x_{53}) - x_{13}(x_{31}d_{55} - a_{35}a_{51}) + x_{15}(x_{31}x_{53} - d_{33}a_{51})$$

$$|F(1,4,5)| = d_{11}(d_{44}d_{55} - a_{45}x_{54}) - x_{14}(x_{41}d_{55} - a_{45}a_{51}) + x_{15}(x_{41}x_{54} - d_{44}a_{51})$$

$$|F(1,2,3,5)| = \begin{matrix} \left[d_{11}\left\{d_{22}(d_{33}d_{55} - a_{35}x_{53}) - x_{23}(x_{32}d_{55} - a_{35}x_{52}) + a_{25}(x_{32}x_{53} - d_{33}x_{52})\right\}\right] \\ -\left[x_{12}\left\{x_{21}(d_{33}d_{55} - a_{35}x_{53}) - x_{23}(x_{31}d_{55} - a_{35}a_{51}) + a_{25}(x_{31}x_{53} - d_{33}a_{51})\right\}\right] \\ +\left[x_{13}\left\{x_{21}(x_{32}d_{55} - a_{35}x_{52}) - d_{22}(x_{31}d_{55} - a_{35}a_{51}) + a_{25}(x_{31}x_{52} - x_{32}a_{51})\right\}\right] \\ -\left[x_{15}\left\{x_{21}(x_{32}x_{53} - d_{33}x_{52}) - d_{22}(x_{31}x_{53} - d_{33}a_{51}) + x_{23}(x_{31}x_{52} - x_{32}a_{51})\right\}\right] \end{matrix}$$

$$|F(1,2,4,5)| = \begin{matrix} \left[d_{11}\left\{d_{22}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{42}d_{55} - a_{45}x_{52}) + a_{25}(x_{42}x_{54} - d_{44}x_{52})\right\}\right] \\ -\left[x_{12}\left\{x_{21}(d_{44}d_{55} - a_{45}x_{54}) - x_{24}(x_{41}d_{55} - a_{45}a_{51}) + a_{25}(x_{41}x_{54} - d_{44}a_{51})\right\}\right] \\ +\left[x_{14}\left\{x_{21}(x_{42}d_{55} - a_{45}x_{52}) - d_{22}(x_{41}d_{55} - a_{45}a_{51}) + a_{25}(x_{41}x_{52} - x_{42}a_{51})\right\}\right] \\ -\left[x_{15}\left\{x_{21}(x_{42}x_{54} - d_{44}x_{52}) - d_{22}(x_{41}x_{54} - d_{44}a_{51}) + x_{24}(x_{41}x_{52} - x_{42}a_{51})\right\}\right] \end{matrix}$$

$$\left|F(1,3,4,5)\right| = \begin{bmatrix} d_{11}\{d_{33}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{43}d_{55}-a_{45}x_{53})+a_{35}(x_{43}x_{54}-d_{44}x_{53})\} \\ -\left[x_{13}\{x_{31}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{41}d_{55}-a_{45}a_{51})+a_{35}(x_{41}x_{54}-d_{44}a_{51})\}\right] \\ +\left[x_{14}\{x_{31}(x_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{41}d_{55}-a_{45}a_{51})+a_{35}(x_{41}x_{53}-x_{43}a_{51})\}\right] \\ -\left[x_{15}\{x_{31}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{41}x_{54}-d_{44}a_{51})+x_{34}(x_{41}x_{53}-x_{43}a_{51})\}\right] \end{bmatrix}$$

Since $x_{ij}=0$

$$\left|F(1,5)\right| = d_{11}d_{55} \geq 0$$

$$\left|F(1,2,5)\right| = d_{11}d_{22}d_{55} \geq 0$$

$$\left|F(1,3,5)\right| = d_{11}d_{33}d_{55} \geq 0$$

$$\left|F(1,4,5)\right| = d_{11}d_{44}d_{55} \geq 0$$

$$\left|F(1,2,3,5)\right| = d_{11}d_{22}d_{33}d_{55} \geq 0$$

$$\left|F(1,2,4,5)\right| = d_{11}d_{22}d_{44}d_{55} \geq 0$$

$$\left|F(1,3,4,5)\right| = d_{11}d_{33}d_{44}d_{55} \geq 0$$

$$\left|F(1,2,3,4,5)\right| = d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$$

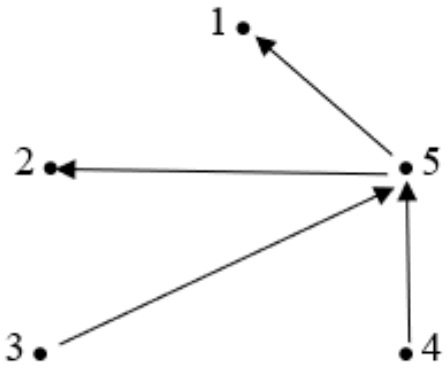

Fig 8:

$$G = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & a_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ a_{51} & a_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

Apart from a_{25} now x_{25} and x_{52} now a_{52} ,all the other entries are exactly those in case 2.7 thus we only consider submatrices that have 2 and 5.

$$\left|G(2,5)\right| = d_{22}d_{55} - x_{25}a_{52}$$

$$\left|G(1,2,5)\right| = d_{11}(d_{22}d_{55}-x_{25}a_{52}) - x_{12}(x_{21}d_{55}-x_{25}a_{51}) + x_{15}(x_{21}a_{52}-d_{22}a_{51})$$

$$\left|G(2,3,5)\right| = d_{22}(d_{33}d_{55}-a_{35}x_{53}) - x_{23}(x_{32}d_{55}-a_{35}a_{52}) + x_{25}(x_{32}x_{53}-d_{33}a_{52})$$

$$\left|G(2,4,5)\right| = d_{22}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{24}\left(x_{42}d_{55} - a_{45}a_{52}\right) + x_{25}\left(x_{42}x_{54} - d_{44}a_{52}\right)$$

$$\left|G(1,2,3,5)\right| = \begin{aligned} &\left[d_{11}\left\{d_{22}\left(d_{33}d_{55} - a_{35}x_{53}\right) - x_{23}\left(x_{32}d_{55} - a_{35}a_{52}\right) + x_{25}\left(x_{32}x_{53} - d_{33}a_{52}\right)\right\}\right] \\ &-\left[x_{12}\left\{x_{21}\left(d_{33}d_{55} - a_{35}x_{53}\right) - x_{23}\left(x_{31}d_{55} - a_{35}a_{51}\right) + x_{25}\left(x_{31}x_{53} - d_{33}a_{51}\right)\right\}\right] \\ &+\left[x_{13}\left\{x_{21}\left(x_{32}d_{55} - a_{35}a_{52}\right) - d_{22}\left(x_{31}d_{55} - a_{35}a_{51}\right) + x_{25}\left(x_{31}a_{52} - x_{32}a_{51}\right)\right\}\right] \\ &-\left[x_{15}\left\{x_{21}\left(x_{32}x_{53} - d_{33}a_{52}\right) - d_{22}\left(x_{3153} - d_{33}a_{51}\right) + x_{23}\left(x_{31}a_{52} - x_{32}a_{51}\right)\right\}\right] \end{aligned}$$

$$\left|G(1,2,4,5)\right| = \begin{aligned} &\left[d_{11}\left\{d_{22}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{24}\left(x_{42}d_{55} - a_{45}a_{52}\right) + x_{25}\left(x_{42}x_{54} - d_{44}a_{52}\right)\right\}\right] \\ &-\left[x_{12}\left\{x_{21}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{24}\left(x_{41}d_{55} - a_{45}x_{51}\right) + x_{25}\left(x_{41}x_{54} - d_{44}x_{51}\right)\right\}\right] \\ &+\left[x_{14}\left\{x_{21}\left(x_{42}d_{55} - a_{45}a_{52}\right) - d_{22}\left(x_{41}d_{55} - a_{45}x_{51}\right) + x_{25}\left(x_{41}a_{52} - x_{42}x_{51}\right)\right\}\right] \\ &-\left[x_{15}\left\{x_{21}\left(x_{42}x_{54} - d_{44}a_{52}\right) - d_{22}\left(x_{41}x_{54} - d_{44}x_{51}\right) + x_{24}\left(x_{41}a_{52} - x_{42}x_{51}\right)\right\}\right] \end{aligned}$$

$$\left|G(2,3,4,5)\right| = \begin{aligned} &\left[d_{22}\left\{d_{33}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{34}\left(x_{43}d_{55} - a_{45}x_{53}\right) + a_{35}\left(x_{43}x_{54} - d_{44}x_{53}\right)\right\}\right] \\ &-\left[x_{23}\left\{x_{32}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{34}\left(x_{42}d_{55} - a_{45}a_{52}\right) + a_{35}\left(x_{42}x_{54} - d_{44}a_{52}\right)\right\}\right] \\ &+\left[x_{24}\left\{x_{32}\left(x_{43}d_{55} - a_{45}x_{53}\right) - d_{33}\left(x_{42}d_{55} - a_{45}a_{52}\right) + a_{35}\left(x_{42}x_{53} - x_{43}a_{52}\right)\right\}\right] \\ &-\left[x_{25}\left\{x_{32}\left(x_{43}x_{54} - d_{44}x_{53}\right) - d_{33}\left(x_{42}x_{54} - d_{44}a_{52}\right) + x_{34}\left(x_{42}x_{53} - x_{43}a_{52}\right)\right\}\right] \end{aligned}$$

$$
\begin{aligned}
\left|G(1,2,3,4,5)\right| = {} & d_{11}\begin{bmatrix} d_{22}\{d_{33}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{43}d_{55}-a_{45}x_{53})+a_{35}(x_{43}x_{54}-d_{44}x_{53})\} \\ -x_{23}\{x_{32}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{42}d_{55}-a_{45}a_{52})+a_{35}(x_{42}x_{54}-d_{44}a_{52})\} \\ +x_{24}\{x_{32}(x_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{42}d_{55}-a_{45}a_{52})+a_{35}(x_{42}x_{53}-x_{43}a_{52})\} \\ -x_{25}\{x_{32}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{42}x_{54}-d_{44}a_{52})+x_{34}(x_{42}x_{53}-x_{43}a_{52})\} \end{bmatrix} \\
& -x_{12}\begin{bmatrix} x_{21}\{d_{33}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{43}d_{55}-a_{45}x_{53})+a_{35}(x_{43}x_{54}-d_{44}x_{53})\} \\ -x_{23}\{x_{31}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{41}d_{55}-a_{45}a_{51})+a_{35}(x_{41}x_{54}-d_{44}a_{51})\} \\ +x_{24}\{x_{31}(x_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{41}d_{55}-a_{45}a_{51})+a_{35}(x_{41}x_{53}-x_{43}a_{51})\} \\ -x_{25}\{x_{31}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{41}x_{54}-d_{44}a_{51})+x_{34}(x_{41}x_{53}-x_{43}a_{51})\} \end{bmatrix} \\
& +x_{13}\begin{bmatrix} x_{21}\{x_{32}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{42}d_{55}-a_{45}a_{52})+a_{35}(x_{42}x_{54}-d_{44}a_{52})\} \\ -d_{22}\{x_{31}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{41}d_{55}-a_{45}a_{51})+a_{35}(x_{41}x_{54}-d_{44}a_{51})\} \\ +x_{24}\{x_{31}(x_{42}d_{55}-a_{45}a_{52})-x_{32}(x_{41}d_{55}-a_{45}a_{51})+a_{35}(x_{41}a_{52}-x_{42}a_{51})\} \\ -x_{25}\{x_{31}(x_{42}x_{54}-d_{44}a_{52})-x_{32}(x_{41}x_{54}-d_{44}a_{51})+x_{34}(x_{41}a_{52}-x_{42}a_{51})\} \end{bmatrix} \\
& -x_{14}\begin{bmatrix} x_{21}\{x_{32}(x_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{42}d_{55}-a_{45}a_{52})+a_{35}(x_{42}x_{53}-x_{43}a_{52})\} \\ -d_{22}\{x_{31}(x_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{41}d_{55}-a_{45}a_{51})+a_{35}(x_{41}x_{53}-x_{43}a_{51})\} \\ +x_{23}\{x_{31}(x_{42}d_{55}-a_{45}a_{52})-x_{32}(x_{41}d_{55}-a_{45}a_{51})+a_{35}(x_{41}a_{52}-x_{42}a_{51})\} \\ -x_{25}\{x_{31}(x_{42}x_{53}-x_{43}a_{52})-x_{32}(x_{41}x_{53}-x_{43}a_{51})+d_{33}(x_{41}a_{52}-x_{42}a_{51})\} \end{bmatrix} \\
& +x_{15}\begin{bmatrix} x_{21}\{x_{32}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{42}x_{54}-d_{44}a_{52})+x_{34}(x_{42}x_{53}-x_{43}a_{52})\} \\ -d_{22}\{x_{31}(x_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{41}x_{54}-d_{44}a_{51})+x_{34}(x_{41}x_{53}-x_{43}a_{51})\} \\ +x_{23}\{x_{31}(x_{42}x_{54}-d_{44}a_{52})-x_{32}(x_{41}x_{54}-d_{44}a_{51})+x_{34}(x_{41}a_{52}-x_{42}a_{51})\} \\ -x_{24}\{x_{31}(x_{42}x_{53}-x_{43}a_{52})-x_{32}(x_{41}x_{53}-x_{43}a_{51})+d_{33}(x_{41}a_{52}-x_{42}a_{51})\} \end{bmatrix}
\end{aligned}
$$

Since $x_{ij}=0$

$$\left|G(2,5)\right| = d_{22}d_{55} \geq 0$$

$$\left|G(1,2,5)\right| = d_{11}d_{22}d_{55} \geq 0$$

$$\left|G(2,3,5)\right| = d_{22}d_{33}d_{55} \geq 0$$

$$\left|G(2,4,5)\right| = d_{22}d_{44}d_{55} \geq 0$$

$$\left|G(1,2,3,5)\right| = d_{11}d_{22}d_{33}d_{55} \geq 0$$

$$\left|G(1,2,4,5)\right| = d_{11}d_{22}d_{44}d_{55} \geq 0$$

$$\left|G(2,3,4,5)\right| = d_{22}d_{33}d_{44}d_{55} \geq 0$$

$$\left|G(1,2,3,4,5)\right| = d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$$

Principle minors all non-negative, the digraph has zero completion into an M_0-matrix.

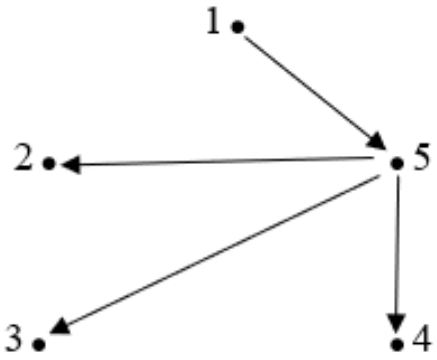

Fig 9:

$$G^T = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & a_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & a_{52} & a_{53} & a_{54} & d_{55} \end{bmatrix}.$$

This digraph is a transpose of the previous digraph hence has zero completion into an M_0 -matrix

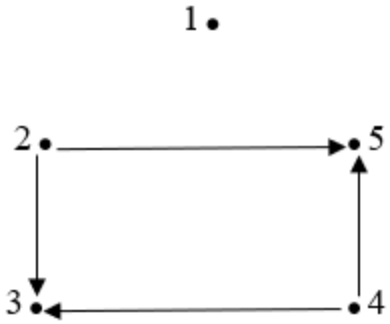

Fig 10:

$$H = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & a_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & a_{43} & d_{44} & a_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

$$|H(1,2)| = d_{11}d_{22} - x_{12}x_{21}$$
$$|H(1,3)| = d_{11}d_{33} - x_{13}x_{31}$$
$$|H(1,4)| = d_{11}d_{44} - x_{14}x_{41}$$
$$|H(1,5)| = d_{11}d_{55} - x_{15}x_{51}$$
$$|H(2,3)| = d_{22}d_{33} - a_{23}x_{32}$$
$$|H(2,4)| = d_{22}d_{44} - x_{24}x_{42}$$
$$|H(2,5)| = d_{22}d_{55} - a_{25}x_{52}$$

$$\left|H\left(3,4\right)\right| = d_{33}d_{44} - x_{34}a_{43}$$

$$\left|H\left(3,5\right)\right| = d_{33}d_{55} - x_{35}x_{53}$$

$$\left|H\left(4,5\right)\right| = d_{44}d_{55} - a_{45}x_{54}$$

$$\left|H\left(1,2,3\right)\right| = d_{11}\left(d_{22}d_{33} - a_{23}x_{32}\right) - x_{12}\left(x_{21}d_{33} - a_{23}x_{31}\right) + x_{13}\left(x_{21}x_{32} - d_{22}x_{31}\right)$$

$$\left|H\left(1,2,4\right)\right| = d_{11}\left(d_{22}d_{44} - x_{24}x_{42}\right) - x_{12}\left(x_{21}d_{44} - x_{24}x_{41}\right) + x_{14}\left(x_{21}x_{42} - d_{22}x_{41}\right)$$

$$\left|H\left(1,2,5\right)\right| = d_{11}\left(d_{22}d_{55} - a_{25}x_{52}\right) - x_{12}\left(x_{21}d_{55} - a_{25}x_{51}\right) + x_{15}\left(x_{21}x_{52} - d_{22}x_{51}\right)$$

$$\left|H\left(1,3,4\right)\right| = d_{11}\left(d_{33}d_{44} - x_{34}a_{43}\right) - x_{13}\left(x_{31}d_{44} - x_{34}x_{41}\right) + x_{14}\left(x_{31}a_{43} - d_{33}x_{41}\right)$$

$$\left|H\left(1,3,5\right)\right| = d_{11}\left(d_{33}d_{55} - x_{35}x_{53}\right) - x_{13}\left(x_{31}d_{55} - x_{35}x_{51}\right) + x_{15}\left(x_{31}x_{53} - d_{33}x_{51}\right)$$

$$\left|H\left(1,4,5\right)\right| = d_{11}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{14}\left(x_{41}d_{55} - a_{45}x_{51}\right) + x_{15}\left(x_{41}x_{54} - d_{44}x_{51}\right)$$

$$\left|H\left(2,3,4\right)\right| = d_{22}\left(d_{33}d_{44} - x_{34}a_{43}\right) - a_{23}\left(x_{32}d_{44} - x_{34}x_{42}\right) + x_{24}\left(x_{32}a_{43} - d_{33}x_{42}\right)$$

$$\left|H\left(2,3,5\right)\right| = d_{22}\left(d_{33}d_{55} - x_{35}x_{53}\right) - a_{23}\left(x_{32}d_{55} - x_{35}x_{52}\right) + a_{25}\left(x_{32}x_{53} - d_{33}x_{52}\right)$$

$$\left|H\left(2,4,5\right)\right| = d_{22}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{24}\left(x_{42}d_{55} - a_{45}x_{52}\right) + a_{25}\left(x_{42}x_{54} - d_{44}x_{52}\right)$$

$$\left|H\left(3,4,5\right)\right| = d_{33}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{34}\left(a_{43}d_{55} - a_{45}x_{53}\right) + x_{35}\left(a_{43}x_{54} - d_{44}x_{53}\right)$$

$$\left|H\left(1,2,3,4\right)\right| = \begin{aligned} &\left[d_{11}\left\{d_{22}\left(d_{33}d_{44} - x_{34}a_{43}\right) - a_{23}\left(x_{32}d_{44} - x_{34}x_{42}\right) + x_{24}\left(x_{32}a_{43} - d_{33}x_{42}\right)\right\}\right] \\ &-\left[x_{12}\left\{x_{21}\left(d_{33}d_{44} - x_{34}a_{43}\right) - a_{23}\left(x_{31}d_{44} - x_{34}x_{41}\right) + x_{24}\left(x_{31}a_{43} - d_{33}x_{41}\right)\right\}\right] \\ &+\left[x_{13}\left\{x_{21}\left(x_{32}d_{44} - x_{34}x_{42}\right) - d_{22}\left(x_{31}d_{44} - x_{34}x_{41}\right) + x_{24}\left(x_{31}x_{42} - x_{32}x_{41}\right)\right\}\right] \\ &-\left[x_{14}\left\{x_{21}\left(x_{32}a_{43} - d_{33}x_{42}\right) - d_{22}\left(x_{31}a_{43} - d_{33}x_{41}\right) + a_{23}\left(x_{31}x_{42} - x_{32}x_{41}\right)\right\}\right] \end{aligned}$$

$$\left|H\left(1,2,3,5\right)\right| = \begin{aligned} &\left[d_{11}\left\{d_{22}\left(d_{33}d_{55} - x_{35}x_{53}\right) - a_{23}\left(x_{32}d_{55} - x_{35}x_{52}\right) + a_{25}\left(x_{32}x_{53} - d_{33}x_{52}\right)\right\}\right] \\ &-\left[x_{12}\left\{x_{21}\left(d_{33}d_{55} - x_{35}x_{53}\right) - a_{23}\left(x_{31}d_{55} - x_{35}x_{51}\right) + a_{25}\left(x_{31}x_{53} - d_{33}x_{51}\right)\right\}\right] \\ &+\left[x_{13}\left\{x_{21}\left(x_{32}d_{55} - x_{35}x_{52}\right) - d_{22}\left(x_{31}d_{55} - x_{35}x_{51}\right) + a_{25}\left(x_{31}x_{52} - x_{32}x_{51}\right)\right\}\right] \\ &-\left[x_{15}\left\{x_{21}\left(x_{32}x_{53} - d_{33}x_{52}\right) - d_{22}\left(x_{31}x_{53} - d_{33}x_{51}\right) + a_{23}\left(x_{31}x_{52} - x_{32}x_{51}\right)\right\}\right] \end{aligned}$$

$$\left|H\left(1,2,4,5\right)\right| = \begin{aligned} &\left[d_{11}\left\{d_{22}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{24}\left(x_{42}d_{55} - a_{45}x_{52}\right) + a_{25}\left(x_{42}x_{54} - d_{44}x_{52}\right)\right\}\right] \\ &-\left[x_{12}\left\{x_{21}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{24}\left(x_{41}d_{55} - a_{45}x_{51}\right) + a_{25}\left(x_{41}x_{54} - d_{44}x_{51}\right)\right\}\right] \\ &+\left[x_{14}\left\{x_{21}\left(x_{42}d_{55} - a_{45}x_{52}\right) - d_{22}\left(x_{41}d_{55} - a_{45}x_{51}\right) + a_{25}\left(x_{41}x_{52} - x_{42}x_{51}\right)\right\}\right] \\ &-\left[x_{15}\left\{x_{21}\left(x_{42}x_{54} - d_{44}x_{52}\right) - d_{22}\left(x_{41}x_{54} - d_{44}x_{51}\right) + x_{24}\left(x_{41}x_{52} - x_{42}x_{51}\right)\right\}\right] \end{aligned}$$

$$\left|H\left(1,3,4,5\right)\right| = \begin{aligned} &\left[d_{11}\left\{d_{33}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{34}\left(a_{43}d_{55} - a_{45}x_{53}\right) + x_{35}\left(a_{43}x_{54} - d_{44}x_{53}\right)\right\}\right] \\ &-\left[x_{13}\left\{x_{31}\left(d_{44}d_{55} - a_{45}x_{54}\right) - x_{34}\left(x_{41}d_{55} - a_{45}x_{51}\right) + x_{35}\left(x_{41}x_{54} - d_{44}x_{51}\right)\right\}\right] \\ &+\left[x_{14}\left\{x_{31}\left(a_{43}d_{55} - a_{45}x_{53}\right) - d_{33}\left(x_{41}d_{55} - a_{45}x_{51}\right) + x_{35}\left(x_{41}x_{53} - a_{43}x_{51}\right)\right\}\right] \\ &-\left[x_{15}\left\{x_{31}\left(a_{43}x_{54} - d_{44}x_{53}\right) - d_{33}\left(x_{41}x_{54} - d_{44}x_{51}\right) + x_{34}\left(x_{41}x_{53} - a_{43}x_{51}\right)\right\}\right] \end{aligned}$$

$$\left|H\left(2,3,4,5\right)\right| = \begin{matrix} \left[d_{22}\left\{d_{33}\left(d_{44}d_{55}-a_{45}x_{54}\right)-x_{34}\left(a_{43}d_{55}-a_{45}x_{53}\right)+x_{35}\left(a_{43}x_{54}-d_{44}x_{53}\right)\right\}\right] \\ -\left[a_{23}\left\{x_{32}\left(d_{44}d_{55}-a_{45}x_{54}\right)-x_{34}\left(x_{42}d_{55}-a_{45}x_{52}\right)+x_{35}\left(x_{42}x_{54}-d_{44}x_{52}\right)\right\}\right] \\ +\left[x_{24}\left\{x_{32}\left(a_{43}d_{55}-a_{45}x_{53}\right)-d_{33}\left(x_{42}d_{55}-a_{45}x_{52}\right)+x_{35}\left(x_{42}x_{53}-a_{43}x_{52}\right)\right\}\right] \\ -\left[a_{25}\left\{x_{32}\left(a_{43}x_{54}-d_{44}x_{53}\right)-d_{33}\left(x_{42}x_{54}-d_{44}x_{52}\right)+x_{34}\left(x_{42}x_{53}-a_{43}x_{52}\right)\right\}\right] \end{matrix}$$

Since $x_{ij}=0$

$$\left|H\left(1,2\right)\right| = d_{11}d_{22} \geq 0$$

$$\left|H\left(1,3\right)\right| = d_{11}d_{33} \geq 0$$

$$\left|H\left(1,4\right)\right| = d_{11}d_{44} \geq 0$$

$$\left|H\left(1,5\right)\right| = d_{11}d_{55} \geq 0$$

$$\left|H\left(2,3\right)\right| = d_{22}d_{33} \geq 0$$

$$\left|H\left(2,4\right)\right| = d_{22}d_{44} \geq 0$$

$$\left|H\left(2,5\right)\right| = d_{22}d_{55} \geq 0$$

$$\left|H\left(3,4\right)\right| = d_{33}d_{44} \geq 0$$

$$\left|H\left(3,5\right)\right| = d_{33}d_{55} \geq 0$$

$$\left|H\left(4,5\right)\right| = d_{44}d_{55} \geq 0$$

$$
\begin{aligned}
\left|H(1,2,3,4,5)\right| = {} & d_{11}\begin{bmatrix} d_{22}\{d_{33}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(a_{43}d_{55}-a_{45}x_{53})+x_{35}(a_{43}x_{54}-d_{44}x_{53})\} \\ -a_{23}\{x_{32}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{42}d_{55}-a_{45}x_{52})+x_{35}(x_{42}x_{54}-d_{44}x_{52})\} \\ +x_{24}\{x_{32}(a_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{42}d_{55}-a_{45}x_{52})+x_{35}(x_{42}x_{53}-a_{43}x_{52})\} \\ -a_{25}\{x_{32}(a_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{42}x_{54}-d_{44}x_{52})+x_{34}(x_{42}x_{53}-a_{43}x_{52})\} \end{bmatrix} \\
& -x_{12}\begin{bmatrix} x_{21}\{d_{33}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(a_{43}d_{55}-a_{45}x_{53})+x_{35}(a_{43}x_{54}-d_{44}x_{53})\} \\ -a_{23}\{x_{31}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{41}d_{55}-a_{45}x_{51})+x_{35}(x_{41}x_{54}-d_{44}x_{51})\} \\ +x_{24}\{x_{31}(a_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{41}d_{55}-a_{45}x_{51})+x_{35}(x_{41}x_{53}-a_{43}x_{51})\} \\ -a_{25}\{x_{31}(a_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{41}x_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{53}-a_{43}x_{51})\} \end{bmatrix} \\
& +x_{13}\begin{bmatrix} x_{21}\{x_{32}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{42}d_{55}-a_{45}x_{52})+x_{35}(x_{42}x_{54}-d_{44}x_{52})\} \\ -d_{22}\{x_{31}(d_{44}d_{55}-a_{45}x_{54})-x_{34}(x_{41}d_{55}-a_{45}x_{51})+x_{35}(x_{41}x_{54}-d_{44}x_{51})\} \\ +x_{24}\{x_{31}(x_{42}d_{55}-a_{45}x_{52})-x_{32}(x_{41}d_{55}-a_{45}x_{51})+x_{35}(x_{41}x_{52}-x_{42}x_{51})\} \\ -a_{25}\{x_{31}(x_{42}x_{54}-d_{44}x_{52})-x_{32}(x_{41}x_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{52}-x_{42}x_{51})\} \end{bmatrix} \\
& -x_{14}\begin{bmatrix} x_{21}\{x_{32}(a_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{42}d_{55}-a_{45}x_{52})+x_{35}(x_{42}x_{53}-a_{43}x_{52})\} \\ -d_{22}\{x_{31}(a_{43}d_{55}-a_{45}x_{53})-d_{33}(x_{41}d_{55}-a_{45}x_{51})+x_{35}(x_{41}x_{53}-a_{43}x_{51})\} \\ +a_{23}\{x_{31}(x_{42}d_{55}-a_{45}x_{52})-x_{32}(x_{41}d_{55}-a_{45}x_{51})+x_{35}(x_{41}x_{52}-x_{42}x_{51})\} \\ -a_{25}\{x_{31}(x_{42}x_{53}-a_{43}x_{52})-x_{32}(x_{41}x_{53}-a_{43}x_{51})+d_{33}(x_{41}x_{52}-x_{42}x_{51})\} \end{bmatrix} \\
& +x_{15}\begin{bmatrix} x_{21}\{x_{32}(a_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{42}x_{54}-d_{44}x_{52})+x_{34}(x_{42}x_{53}-a_{43}x_{52})\} \\ -d_{22}\{x_{31}(a_{43}x_{54}-d_{44}x_{53})-d_{33}(x_{41}x_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{53}-a_{43}x_{51})\} \\ +a_{23}\{x_{31}(x_{42}x_{54}-d_{44}x_{52})-x_{32}(x_{41}x_{54}-d_{44}x_{51})+x_{34}(x_{41}x_{52}-x_{42}x_{51})\} \\ -x_{24}\{x_{31}(x_{42}x_{53}-a_{43}x_{52})-x_{32}(x_{41}x_{53}-a_{43}x_{51})+d_{33}(x_{41}x_{52}-x_{42}x_{51})\} \end{bmatrix}
\end{aligned}
$$

$$\left|H(1,2,3)\right| = d_{11}d_{22}d_{33} \geq 0$$

$$\left|H(1,2,4)\right| = d_{11}d_{22}d_{44} \geq 0$$

$$\left|H(1,2,5)\right| = d_{11}d_{22}d_{55} \geq 0$$

$$\left|H(1,3,4)\right| = d_{11}d_{33}d_{44} \geq 0$$

$$\left|H(1,3,5)\right| = d_{11}d_{33}d_{55} \geq 0$$

$$\left|H(1,4,5)\right| = d_{11}d_{44}d_{55} \geq 0$$

$$\left|H(2,3,4)\right| = d_{22}d_{33}d_{44} \geq 0$$

$$\left|H(2,3,5)\right| = d_{22}d_{33}d_{55} \geq 0$$

$$\left|H(2,4,5)\right| = d_{22}d_{44}d_{55} \geq 0$$

$$\left|H(3,4,5)\right| = d_{33}d_{44}d_{55} \geq 0$$

$$\left|H(1,2,3,4)\right| = d_{11}d_{22}d_{33}d_{44} \geq 0$$

$$\left|H(1,2,3,5)\right| = d_{11}d_{22}d_{33}d_{55} \geq 0$$

$$\left|H\left(1,2,4,5\right)\right| = d_{11}d_{22}d_{44}d_{55} \geq 0$$
$$\left|H\left(1,3,4,5\right)\right| = d_{11}d_{33}d_{44}d_{55} \geq 0$$
$$\left|H\left(2,3,4,5\right)\right| = d_{22}d_{33}d_{44}d_{55} \geq 0$$
$$\left|H\left(1,2,3,4,5\right)\right| = d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$$

Principle minors all non-negative, the digraph has zero completion into an M_0 -matrix.

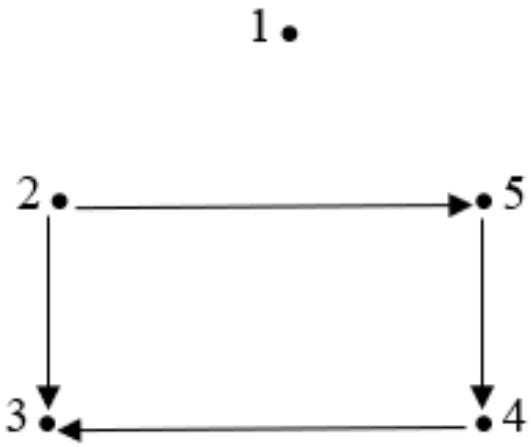

Fig 11:

$$J = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & a_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & a_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & a_{54} & d_{55} \end{bmatrix}$$

Apart from a_{45} now x_{45} and x_{54} now a_{54} ,all the other entries are similar to those in Case 2.10 thus we only consider submatrices that have 4 and 5.

All are non-negative so there is a zero completion into an M_0 -matrix for the digraph.

5. Di-graphs without Mo – Completion

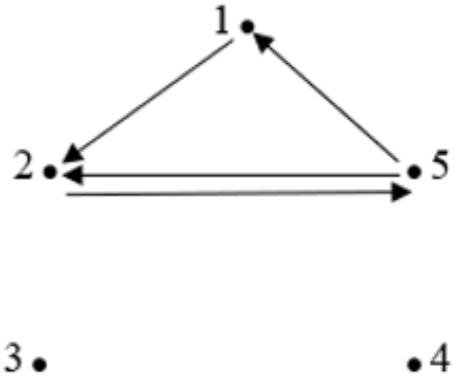

Fig 12:

$$A=\begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & a_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ a_{51} & a_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

We show that A has no M_0 -completion Let

$$A=\begin{bmatrix} 1 & -2 & x & d & z \\ f & 2 & v & k & -1 \\ h & s & 1 & b & m \\ n & r & g & 1 & p \\ -1 & -2 & v & w & 1 \end{bmatrix}$$

$$|A(1,2)|_{=} 2+2f$$
$$|A(1,3)|_{=} 1-hx$$
$$|A(1,4)|_{=} 1-dn$$
$$|A(1,5)|_{=} 1+z$$
$$|A(2,3)|_{=} 2-vs$$
$$|A(2,4)|_{=} 2-rk$$
$$|A(2,5)|_{=} 0$$
$$|A(3,4)|_{=} 1-gb$$
$$|A(4,5)|_{=} 1-pw$$
$$|A(1,2,5)|_{=} 1(0)+2(f-1)+z(-2f+2)$$
$$=-2+2f+2z-2zf<0 \text{ ,since } f<0, z<0$$

which proves that the digraph has no M_0 -completion.

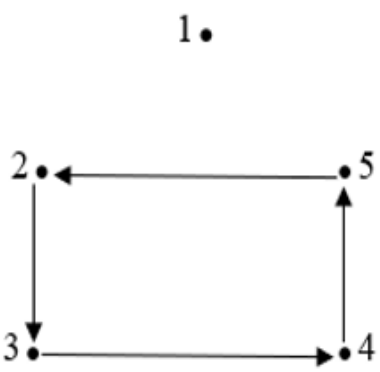

Fig 13:

$$B=\begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & a_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ a_{51} & a_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

We show that B has no M_0 -completion Let

$$B=\begin{bmatrix} 1 & b & x & g & r \\ m & 1 & -2 & k & v \\ z & f & 2 & -3 & w \\ n & h & y & 0 & -1 \\ j & -1 & d & s & 1 \end{bmatrix}$$

$|B(1,2)| = 1-bm$

$|B(1,3)| = 2-xz$

$|B(1,4)| = 0-ng = -ng < 0$ since $g \le 0, n \le 0$

$|B(2,3)| = 2+2f$

$|B(2,4)| = 0-hk = -hk < 0$ since $h \le 0, k \le 0$

Therefore, there is no M_0 -completion for the digraph.

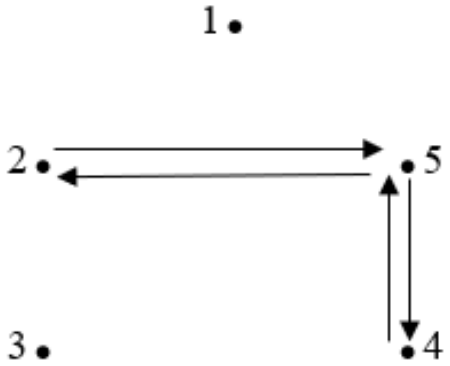

Fig 14:

$$C=\begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & a_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ x_{51} & a_{52} & x_{53} & a_{54} & d_{55} \end{bmatrix}$$

We show that it has no M_0 -completion Let

$$C=\begin{bmatrix} 1 & p & w & y & t \\ x & 1 & v & m & -1 \\ s & r & 1 & d & e \\ h & a & g & 2 & -2 \\ f & -2 & j & -1 & 2 \end{bmatrix}$$

$$|C(1,2)| = 1 - px$$
$$|C(1,3)| = 1 - sw$$
$$|C(1,4)| = 2 - hy$$
$$|C(1,4)| = 2 - ft$$
$$|C(2,3)| = 1 - rv$$
$$|C(2,4)| = 2 - am$$
$$|C(2,5)| = 0$$
$$|C(3,4)| = 2 - dg$$
$$|C(4,5)| = 2$$
$$|C(2,4,5)| = 1(4-2) - m(2a-4) - 1(-a+4)$$
$$= (-2 - 2am + 4m + a) < 0 \text{ since } a \le 0, m \le 0$$

Therefore, there is no M_0 -completion for this digraph.

6. Conclusion

This study investigated non-isomorphic digraphs with five vertices and four arcs and established that all associated partial matrices that are not cycles admits M_0 -completion whereas those corresponding to cyclic digraphs do not.
Further research can be done to verify whether the same conclusion applies to matrices of order 5 x 5 with more than 4 edges and also matrices of higher order, that is, 6 x 6 and above.

References

1. Choi JY, DeAlba LM, Hogben L, Maxwell M, Wangness A. The P0-matrix completion problem. Electron J Linear Algebra. 2002;9:1–20.
2. DeAlba LM, Hogben L, Samar BK. The Q-matrix completion problem. Electron J Linear Algebra. 2009;18:176–191.
3. Hogben L, Kivunge B, Nordstrom SK, Shedenhelm M. The nonnegative P0-matrix completion problem. Electron J Linear Algebra. 2003;10:46–59.
4. Hogben L. Graph theoretic methods for matrix completion problems. Linear Algebra Appl. 2001;328:161–202.
5. Hogben L, Wangness A, Snider. Completion problems for various classes of P-matrices. Linear Algebra Appl. 2006;413:342–354.
6. Horn RA, Johnson CR. Matrix Analysis. Cambridge: Cambridge University Press; 1985.
7. Johnson CR. Matrix completion problem: a survey. Proc Sympos Appl Math. 1990;40:171–198.
8. Johnson CR, Smith RL. Completion of partial matrices with M-matrix constraints. Linear Algebra Appl. 1991;147:201–211.
9. Johnson CR, Smith RL. The symmetric inverse M-matrix completion problem. Linear Algebra Appl. 1999;290:193–212.
10. Mutembei J. M0-matrix completion problem for digraphs of 2×2 and 3×3 matrices [master's thesis]. Nairobi: Kenyatta University; 2007.
11. Kabusia R. M0-matrix completion problem for digraphs of 4×4 matrices for q = 0, 1, 2 and 3 [master's thesis]. Nairobi: Kenyatta University; 2008.
12. Grone R, Johnson CR, Sa EM, Wolkowicz H. Positive definite completions of partial Hermitian matrices. Linear Algebra Appl. 1984;58:109–124.

How to Cite This Article

Oduor G, Okelo B. On admission of -completion by non-isomorphic digraphs and their associated partial matrices. International Journal of Applied Mathematics and Numerical Research. 2026;2(3):1–27.

Creative Commons (CC) License