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A Class of Symmetric Trigonometrically-Fitted Hybrid Method for Second-Order Initial Value Problems

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Abstract
This paper develops a new class of symmetric hybrid linear multistep method for solving special second-order initial value problems in ordinary differential equations. The proposed method has been derived using trigonometrically fitted basis function, achieving order six accuracy and minimal local truncation errors. The boundary locus presented illustrates the stability region of the method. Numerical experiments on test problems confirms the applicability of the proposed method.

Keywords: Symmetric, Hybrid LMMs, p-stability, Basis function, convergence

Introduction

Our interest in this paper is the numerical solution of second-order initial value problems (IVPs) in ordinary differential equations (ODEs) given by

$$\begin{cases} y'' = f\big(x, y(x)\big), \quad x \in [x_0, X], \\ y(x_0) = y_0, \\ y'(x_0) = y_0', \end{cases}$$

where $f: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is assumed to be sufficiently smooth to ensure the existence and uniqueness of the solution. The major challenge with problem [Sode] is that the first derivative does not appear explicitly, and their solutions are kownn to exhibit oscillatory behaviour. For example, such problems are often encountered in orbital dynamics, quantum mechanics, and celestial mechanics, just to mention a few. Several numerical methods have been designed over the years to address the inherent stiff and oscillatory nature of problem [Sode]. There is a vast literature on classical linear multistep methods for solving the problem [Sode], but their performance is fundamentally limited for oscillatory solutions. Lambert and Watson showed that no p-stable multistep method can achieve an algebraic order greater than two, regardless of the step size. This shortcoming has led many mathematicians to the development of extended frameworks that go beyond LMMs. Among these, hybrid linear multistep methods where one or more off-step points are incorporated into the LMM to improve the order of accuracy have proven very effective. See, and the references therein.

Another important consideration along this direction is the symmetric property of the numerical scheme. Symmetric formulas tend to preserve qualitative properties of the exact solution, especially in Hamiltonian or near-Hamiltonian systems. They suppress amplified error and maintain bounded oscillations over a long time integration. By combining high algebraic order and symmetry, hybrid methods achieve both stability and accuracy, making them a powerful tool for solving oscillatory second-order ODEs.

In this work, we aim to develop a new class of symmetric hybrid linear multistep method via trigonometrically-fitted basis

function for accurately integrating a family of problems described by [Sode].
The theory of Linear Multistep Method
The k-step Linear Multistep Methods (LMMs) for solving the initial value problem [Sode] numerically is given by

$$\sum_{j=0}^{k} \alpha_j \, y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j \, f_{n+j}, \quad k \geq 2$$

where it is assumed that $\alpha_k = 1$ and $\beta_k \neq 1$. For simplicity, the LMM [LMM] can be written as $\rho(E)y_n = h^2 \sigma E f_n$, where the shift operator is $E^j y_n = yn + j, \quad j \in \mathbb{Z}$. The first and second characteristic polynomials are $\rho(\xi) = \sum_{j=0}^{k} \alpha_j \, \xi^j$ and $\sigma(\xi) = \sum_{j=0}^{k} \beta_j \, \xi^j$, respectively.
To bypass the order barrier imposed on LMMs [LMM], Hybrid methods incorporating additional off-step points have been introduced to construct high order hybrid methods for the direct numerical solution of [Sode]. For such methods, see and the references therein.
Let the corresponding difference operator for the LMM [LMM] be defined by

$$L[y(x); h] = \sum_{j=0}^{k} \left[\alpha_j y(x + jh) - h^2 \beta_j y''(x + jh)\right],$$

where $y(x)$ is an arbitrary continuously differentiable function on the the interval $[a, b]$. Without loss of generality, the Taylor expansion of $y(x)$ about the point x yields

$$\sum_{j=0}^{k} \left[\alpha_j y(x + jh) - h^2 \beta_j y''(x + jh)\right] = C_{p+2} h^{p+2} y^{(p+2)}(x) + O\left(h^{(p+3)}\right),$$

with an error constant given by

$$C_q = \frac{1}{q!} \sum_{j=0}^{k} j^{q-2} \left(j^2 \alpha_j - q(q-1)\beta_j\right) - \sum_{j=0}^{k} \frac{j^{q-2}}{(q-2)!} \beta_j, \quad q > 2.$$

From [error constant], C_{p+2} is the error constant, $C_{p+2} h^{p+2} y^{(p+2)}(x)$ represent the LTE at the point x_n and p is the algebraic order of the method. The application of method [LMM] on the scalar test equation [test] (see),

$$y'' + \lambda^2 y = 0, \quad \lambda \in \mathbb{R}$$

will yield the finite difference equation of the form

$$\sum_{r=0} A_r \, (\xi^2) y_{n+r} = 0, \quad \xi = i\lambda h,$$

where $A_r(\xi^2)$ are polynomials in ξ^2. It follows that the characteristic equation is

$$\Phi(R, \xi) = \sum_{j=0}^{k} A_r \, (\xi^2) R^2, \quad k > 1.$$

which motivates the following definitions
Definition 1. (Zero-stability) The method [LMM] is zero-stable if the roots of the first charateristic polynomial has modulus less than or equal to one, and if every root of modulus one has multiplicity not greater than two.
Definition 2. (Consistency) The LMM [LMM] is consistent if it has order $\mathrm{p} \geq 1$.
Definition 3. (Convergence) The LMM [LMM] is convergent if it is zero-stable and consistent.
Definition 4. (Interval of Periodicity) The LMM [LMM] is said to have periodicity interval $(0, \mathrm{H}^2)$ if for all $\mathrm{H}^2 \in (0, \mathrm{H}^2)$, the roots of the characteristic polynomial have at most two compex conjugate roots of modulus one while other roots are less than or equal to one.
Definition 5. (P-stability) The LMM [LMM] is p-stable if $(0, \mathrm{H}^2) = (0, \infty)$.
Construction of trigonometriclly-fitted symmetric hybrid linear multistep method
We consider the symmetric hybrid linear multistep method for the numerical integration of [Sode]

$$y(x_n + th) = \sum_{j=0}^{k-1} \lambda_j \, (u) y_{n-j} + h^2 \sum_{j=1}^{k} \beta_{c_j} \, f_{n+c_j} + h^2 [\beta_0(u) f_n + \beta_1(u)(f_{n+1} + f_{n-1})],$$

For $k = 2, \ \lambda_0 = 2, \ \lambda_1 = -1$ the corresponding hybrid methods of interest in this current work take the form

$$y_{n+c_1} = \alpha_2(u) y_{n+1} + \alpha_1(u) y_n + \alpha_0(u) y_{n-1} + h^2 \left[(\phi_0(u) f_n + \phi_1(u) f_{n+c_1}\right],$$

And

$$y_{n+c_2} = \hat{\alpha}_2(u) y_{n+1} + \hat{\alpha}_1(u) y_n + \hat{\alpha}_0(u) y_{n-1} + h^2 \left[(\hat{\phi}_0(u) f_n + \hat{\phi}_1(u) f_{n+c_2}\right].$$

Suppose $u = \omega h$. The coefficients $\beta_j(u)$, $\beta_{c_s}(u)$, $\alpha_l(u)$, and $\phi_j(u)$, with $j = 0,1$ and α_l, $l = 0,1,2$, are unknown functions of the frequency u and the stepsize h. The numerical solution at shifted nodes is denoted by $y_{n\pm c_j} \approx y\left(x_{n\pm c_j}\right)$ and the corresponding function evaluations are $f_{n\pm c_j} = f\left(x_{n\pm c_j}, y_{n\pm c_j}\right)$, $j = 0,1$ and $f_{n\pm j} = f\left(x_{n\pm j}, y_{n\pm j}\right)$, $j = 0,1$.
We define

$$U(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 sin(wx) + a_5 cos(wx)$$

and its second derivative

$$U''(x) = 2a_2 + 6a_3 x - w^2\left(a_4 sin(wx) + a_5 cos(wx)\right)$$

Upon substituting the trigonometrically fitted basis function [basisfunction3] into the collocation equations allows us to solve for the unknown coefficients. Once determined, these coefficients define a trigonometrically fitted hybrid collocation method that is consistent with the underlying oscillatory problem. Now we assume that the interpolating basis function [basisfunction3] coincides with the exact solution at the point $x_{n\pm j}$ to get

$$U\left(x_{n\pm j}\right) = y_{n\pm j}, \quad j = 0,1.$$

Also, collocating [basisderivative3] at x_{n+c_j}, $x_{n\pm j}$, $j = 0,1$ we obtain

$$U''\left(x_{n+c_j}\right) = f_{n+c_j}, \quad U''\left(x_{n\pm j}\right) = f_{n\pm j},\ j = 0,1.$$

Observe that [interpolant] and [collocant] result to a system of six equations with six unknowns imposed by symmetry. Solving this resulting system of equation for $a_j, j = 0(1)5$ and substituting the resulting values of a_j's with $x = x_{n+1} + th$ (without loss of generality, we set $x_n = 0$ so that $x = h + ht$ into [basisfunction3], we obtain a continuous hybrid method with continuous coefficients which is then used to generate the output method and its corresponding hybrid methods. It follows that for $t = 1$ in the continuous method and by carefully choosing $c = \left[-\frac{1}{2}, \frac{1}{2}\right]^T$ we obtain

$$\beta_0(u) = \frac{(s_3/2 - 3s_1/2)\left(u^2 + 2c - 2\right)}{u^2[-s_1 - 3/2(s_1 - s_3 + 2s_2) + s_3 + 1/2(s_3 - s_5 + 2s_2) - 2s_2]},$$

$$\beta_1(u) = \frac{3/2\left(4s_1^2(s_1 + 2s_2) - u^2 s_3\right) + 1/2\left(u^2 s_5 - 4s_1^2(s_3 + 2s_2)\right) + u^2 s_2 - 2s_1(c-1) + 2s_3(c-1)}{u^2[s_1 + 3/2(s_1 - s_3 + 2s_2) - s_3 + 1/2(-s_3 + s_5 - 2s_2) + 2s_2]},$$

$$\beta_{c_1}(u) = \frac{(s_3 - 3s_2/2)\left(u^2 + 2c - 2\right)}{u^2[-s_1 - 3/2(s_1 - s_3 + 2s_2) + s_3 + 1/2(s_3 - s_5 + 2s_2) - 2s_2]},$$

$$\beta_{c_2}(u) = \frac{(s_1 - s_2/2)\left(u^2 + 2c - 2\right)}{u^2[s_1 + 3/2(s_1 - s_3 + 2s_2) - s_3 + 1/2(-s_3 + s_5 - 2s_2) + 2s_2]},$$

$$\phi_0(u) = \frac{3/2\left(4s_1^2(s_1 + 2s_2) - u^2 s_3\right) + 1/2\left(u^2 s_5 - 4s_1^2(s_3 + 2s_2)\right) + u^2 s_2 - 2s_1(c-1) + 2s_3(c-1)}{u^2[s_1 + 3/2(s_1 - s_3 + 2s_2) - s_3 + 1/2(-s_3 + s_5 - 2s_2) + 2s_2]},$$

$$\phi_1(u) = \frac{(s_3/2 - 3s_1/2)\left(u^2 + 2c - 2\right)}{u^2[-s_1 - 3/2(s_1 - s_3 + 2s_2) + s_3 + 1/2(s_3 - s_5 + 2s_2) - 2s_2]}$$

$$\hat{\phi}_0(u) = \frac{3/2\left(4s_1^2(s_1 + 2s_2) - u^2 s_3\right) + 1/2\left(u^2 s_5 - 4s_1^2(s_3 + 2s_2)\right) + u^2 s_2 - 2s_1(c-1) + 2s_3(c-1)}{u^2[s_1 + 3/2(s_1 - s_3 + 2s_2) - s_3 + 1/2(-s_3 + s_5 - 2s_2) + 2s_2]},$$

$$\hat{\phi}_1(u) = \frac{(s_3/2 - 3s_1/2)\left(u^2 + 2c - 2\right)}{u^2[-s_1 - 3/2(s_1 - s_3 + 2s_2) + s_3 + 1/2(s_3 - s_5 + 2s_2) - 2s_2]}$$

$$\alpha_2(u) = \frac{s_1 - \frac{1}{2}s_2}{\Delta(u)}, \quad \alpha_1(u) = \frac{s_3 - s_1}{\Delta(u)}, \quad \alpha_0(u) = \frac{s_1 - \frac{1}{2}s_2}{\Delta(u)}.$$

$\phi_i(u) = \hat{\phi}_i(u), i = 0,1$. And

$$\alpha_i(u) = \hat{\alpha}_i(u), \quad i = 0,1,2.$$

where

$$\Delta(u) = s_1 + \frac{3}{2}(s_1 - s_3 + 2s_2) - s_3 + \frac{1}{2}(-s_3 + s_5 - 2s_2) + 2s_2,, s_k := \sin\frac{ku}{2},\ c := \cos u.$$

Error Analysis and Stability

Local Truncation Error

For small values of u, the above coefficients are subject to heavy cancellation; hence, we employ the Taylor series expansion. See and the references therein. Thus, the coefficients of the main method can be expressed as follows;

$$\beta_0(u) = \frac{13}{60} - \frac{91}{5400}u^2 + \frac{107}{13500}u^4 - \frac{231}{907200}u^6 + \frac{21241}{45360000}u^8 + \mathcal{O}(u^{10})$$

$$\beta_1(u) = \frac{1}{60} - \frac{87}{5400}u^2 + \frac{131}{453600}u^4 - \frac{213}{2721600}u^6 + \frac{21423}{408240000}u^8 + \mathcal{O}(u^{10})$$

$$\beta_{c_1}(u) = \frac{4}{15} - \frac{87}{5400}u^2 + \frac{21241}{45360000}u^4 - \frac{110043}{1360800000}u^6 + \frac{1029677}{459817728000}u^8 + \mathcal{O}(u^{10})$$

$$\beta_{c_2}(u) = \frac{4}{15} - \frac{87}{5400}u^2 + \frac{21241}{45360000}u^4 - \frac{110043}{1360800000}u^6 + \frac{1029677}{459817728000}u^8 + \mathcal{O}(u^{10})$$

Moreover, the coefficients in the corresponding hybrids are

$$\phi_0(u) = \frac{1}{32} - \frac{17}{5400}u^2 + \frac{131}{453600}u^4 - \frac{213}{2721600}u^6 + \frac{21423}{4082400000}u^8 + \mathcal{O}(u^{10})$$

$$\phi_1(u) = -\frac{1}{8} - \frac{11}{5400}u^2 + \frac{27}{13500}u^4 - \frac{111}{907200}u^6 + \frac{10241}{45360000}u^8 + \mathcal{O}(u^{10})$$

$$\alpha_2(u) = \frac{15}{32} - \frac{13}{5400}u^2 - \frac{101}{907200}u^4 - \frac{117}{27216000}u^6 - \frac{2121}{4082400000}u^8 + \mathcal{O}(u^{10})$$

$$\alpha_1(u) = \frac{9}{16} - \frac{17}{5400}u^2 + \frac{131}{453600}u^4 - \frac{103}{2721600}u^6 + \frac{21423}{4082400000}u^8 + \mathcal{O}(u^{10})$$

$$\alpha_0(u) = -\frac{1}{32} - \frac{13}{5400}u^2 - \frac{101}{907200}u^4 - \frac{117}{27216000}u^6 - \frac{2121}{4082400000}u^8 + \mathcal{O}(u^{10})$$

and

$$\hat{\phi}_0(u) = \frac{1}{32} - \frac{17}{5400}u^2 + \frac{131}{453600}u^4 - \frac{213}{2721600}u^6 + \frac{21423}{4082400000}u^8 + \mathcal{O}(u^{10})$$

$$\hat{\phi}_1(u) = -\frac{1}{8} - \frac{11}{5400}u^2 + \frac{27}{13500}u^4 - \frac{111}{907200}u^6 + \frac{10241}{45360000}u^8 + \mathcal{O}(u^{10})$$

$$\hat{\alpha}_2(u) = \frac{15}{32} - \frac{13}{5400}u^2 - \frac{101}{907200}u^4 - \frac{117}{27216000}u^6 - \frac{2121}{4082400000}u^8 + \mathcal{O}(u^{10})$$

$$\hat{\alpha}_1(u) = \frac{9}{16} - \frac{17}{5400}u^2 + \frac{131}{453600}u^4 - \frac{103}{2721600}u^6 + \frac{21423}{4082400000}u^8 + \mathcal{O}(u^{10})$$

$$\hat{\alpha}_0(u) = -\frac{1}{32} - \frac{13}{5400}u^2 - \frac{101}{907200}u^4 - \frac{117}{27216000}u^6 - \frac{2121}{4082400000}u^8 + \mathcal{O}(u^{10})$$

respectively. Thus, the associated local truncation errors are

$$y_{n+1} = -\frac{1}{120960}\, h^8\, y^{(8)}(x_n) + O(h^{10}),$$

and

$$y_{n\pm\frac{1}{2}} = -\frac{1}{768}\, h^5\, y^{(5)}(x_n) + O(h^5).$$

Stability

The application of methods [order 8] and corresponding hybrids [hybrid], [hybrid2] on the test equation [test] helps us to assess the stability property of the new method.

Thus,

$$f_n = -\omega^2 y_n = -\frac{z^2}{h^2}y_n, \quad f_{n\pm\frac{1}{2}} = -\frac{z^2}{h^2}y_{n\pm\frac{1}{2}}, \quad z = \omega h.$$

so that [hybrid] gives

$$y_{n+\frac{1}{2}} = \frac{15}{32}y_{n+1} + \frac{9}{16}y_n - \frac{1}{32}y_{n-1} - \frac{z^2}{32}y_n + \frac{z^2}{8}y_{n+\frac{1}{2}}.$$

Hence

$$\left(1 - \frac{z^2}{8}\right)y_{n+\frac{1}{2}} = \frac{15}{32}y_{n+1} + \left(\frac{9}{16} - \frac{z^2}{32}\right)y_n - \frac{1}{32}y_{n-1}.$$

Therefore

$$y_{n+\frac{1}{2}} = \frac{15y_{n+1} + (18 - z^2)y_n - y_{n-1}}{4(8 - z^2)}.$$

By symmetry the same expression holds for $y_{n-\frac{1}{2}}$. Substituting [eq:half_value_final] into [order 8] yields, after algebraic simplification, a symmetric recurrence of the form

$$y_{n+1} - 2R(z^2)y_n + y_{n-1} = 0.$$

Assuming $y_n = \rho^n$, then the characteristic equation can be written as

$$\rho^2 - 2R(z^2)\rho + 1 = 0,$$

with roots

$$\rho = R(z^2) \pm \sqrt{R(z^2)^2 - 1}.$$

Hence, the recurrence [eq:symmetric_recurrence_corrected] is symmetric and $|R(z^2)| < 1$, the characteristic roots satisfy $|\rho| \leq 1$ for all $z \geq 0$. The boundary locus plot is given in Figure 1

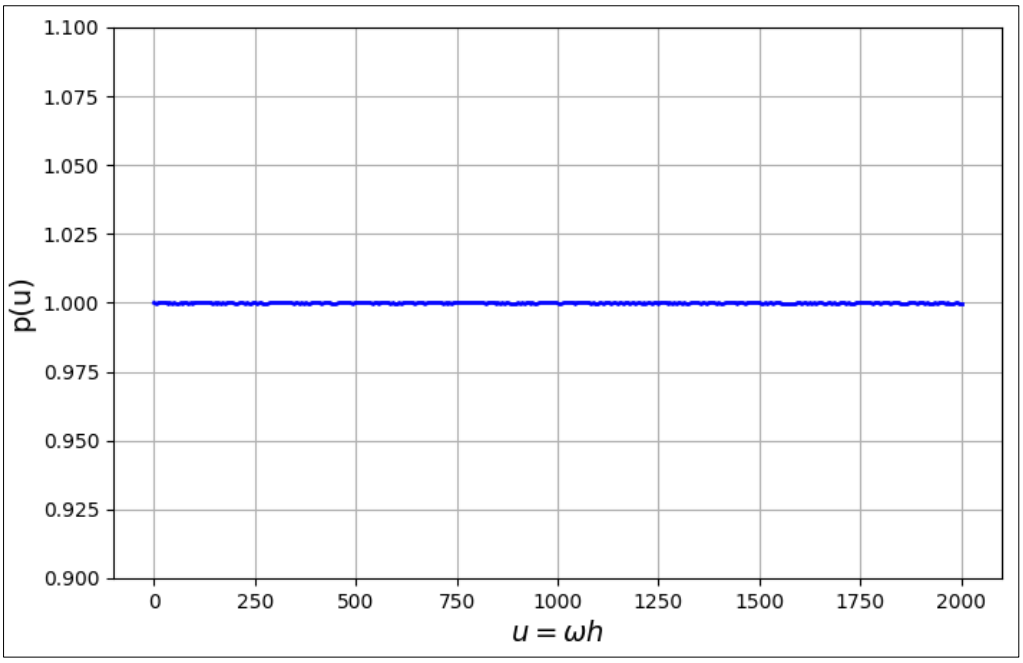

Fig 1: p-stability plot

Numerical Experiment and Discussion

In this section, we assess the performance of the proposed method. Consider the following test problems

Problem 1. *We consider the following orbital problem. See* $\begin{aligned} y'' &= -y + 0.001e^{it}, \\ y(0) &= 1, \quad y'(0) = 0.9995i, \quad i^2 = -1. \end{aligned}$ *whose exact solution is given by*

$$\begin{aligned} y(t) &= u(t) + v(t), \\ u(t) &= cos(t) + 0.0005t\sin(t), \\ v(t) &= sin(t) - 0.0005t\cos(t). \end{aligned}$$

representing a motion on a perturbed circular orbit in the complex plane in which the point $y(t)$ slowly spirals outward such that its distance from the origin at any given time t is described by

$$\Phi(t) = \sqrt{u^2(t) + v^2(t)}$$

Using the predictor $y_{n+2} - 2y_{n+1} + y_n = h^2 f_{n+1}$, we approximate the problem [almost periodic problem] in the interval $[0,40]$ which corresponds to 20 orbits of the points $y(t)$ on the uniform step size

$$h = \frac{\pi}{2^k}, \quad k = 3(1)10$$

The numerical results is presented in Table 1

Table 1: Numerical results for the stiff oscillatory problem [almost periodic problem] at $\mathrm{T} = 40$

k	h	Numerical Solution	Exact Solution	Absolute Error
6	4.908738521234e-02	1.0018912875586	1.0019719765345	8.068897590000e-05
7	2.454369260617e-02	1.0018123538494	1.0019719765345	1.596226851000e-04
8	1.227184630309e-02	1.0019066904601	1.0019719765345	6.528607440000e-05
9	6.135923151543e-03	1.0019951777141	1.0019719765345	2.320117960000e-05
10	3.067961575771e-03	1.0019962610851	1.0019719765345	2.428455060000e-05
11	1.533980787886e-03	1.0019968810935	1.0019719765345	2.490455900000e-05
12	7.669903939428e-04	1.0019971984405	1.0019719765345	2.522190600000e-05
13	3.834951969714e-04	1.0019998965810	1.0019719765345	2.792004650000e-05
14	1.917475984857e-04	1.0019999405781	1.0019719765345	2.796404360000e-05

Problem 2. *We consider the highly stiff oscillatory problem with a more complex pattern. See*

$$y''(t) + \omega^2 y(t) = 8\left(\cos(t) + \frac{2}{3}\cos(3t)\right), \quad y(0) \quad = 1, \quad y'(0) = 0, \quad \text{with} \quad \omega = 5.$$

We integrate the problem in the interval $[0\ 20\pi]$. *The exact solution of the stiff problem [stiff oscillatory problem] is*

$$y(t) = \frac{1}{3}\left(\cos(t) + \cos(3t) + \cos(5t)\right)$$

The numerical results is displayed in Table 2

Table 2: Numerical results for the stiff oscillatory problem [stiff oscillatory problem] at $\mathrm{T} = 20\pi$

k	h	Numerical Solution	Exact Solution	Absolute Error
6	4.908738521234e-02	0.981268798070	1.000000000000	1.873120193000e-02
7	2.454369260617e-02	0.999594333715	1.000000000000	4.056662850000e-04
8	1.227184630309e-02	0.999776643983	1.000000000000	2.233560170000e-04
9	6.135923151543e-03	0.999897836215	1.000000000000	1.021637850000e-04
10	3.067961575771e-03	0.999987546795	1.000000000000	1.245320500000e-05
11	1.533980787886e-03	0.999989653876	1.000000000000	1.034612400000e-05
12	7.669903939428e-04	1.000000673121	1.000000000000	6.731210000000e-07
13	3.834951969714e-04	1.000000042190	1.000000000000	4.219000000000e-08
14	1.917475984857e-04	1.000000022014	1.000000000000	2.201400000000e-08

Conclusion

In this paper, we have developed a new class of symmetric hybrid linear multistep methods for the direct integration of special second-order initial value problems. The construction is based on a trigonometrically fitted basis that incorporates the dominant oscillatory behaviour of the underlying solution. By embedding off-step collocation points and enforcing symmetry, the proposed scheme overcomes the classical order barrier associated with linear multistep methods, having good accuracy with $P = 6$. We have equally presented numerical examples on periodic orbital problem and a highly stiff oscillatory problem. The results obtained shows the efficiency of the proposed method.

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