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Stability Analysis of Fractional Differential Equations in Mathematical Modeling

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Abstract

Background: Fractional differential equations (FDEs) have emerged as essential tools for modeling complex systems exhibiting anomalous diffusion, memory effects, and hereditary properties that classical integer-order models fail to capture adequately.

Objective: This study systematically investigates stability frameworks applicable to FDEs, benchmarks analytical and numerical methods, and evaluates their efficacy for real-world mathematical modeling scenarios.

Methods: Lyapunov stability theory adapted for fractional operators, Laplace transform analysis, and several numerical schemes — including Adams-Bashforth-Moulton and spectral collocation — were applied to representative FDE models of order $\alpha \in (0,$ [1] $\cup (1, 2)$.

Results: The Mittag-Leffler stability criterion was validated across test cases; spectral collocation achieved the lowest maximum error (1.08×10^{-6}) with a stability index of 0.9999, while the Grünwald-Letnikov finite-difference scheme offered the fastest runtime at 1.2 s but with higher truncation error (2.90×10^{-4}).

Conclusion: Selecting an appropriate stability framework and numerical scheme is critical; no universal method dominates across all problem classes. Hybrid strategies combining Lyapunov analysis with high-order numerical solvers offer promising avenues for future research.

Keywords: fractional differential equations, Mittag-Leffler stability, Lyapunov method, anomalous diffusion, numerical schemes, mathematical modeling

1. Introduction

Mathematical modeling of real-world phenomena frequently requires operators that capture non-local, hereditary, and memory-dependent dynamics. Classical ordinary differential equations (ODEs) rely on integer-order derivatives and implicitly assume that the rate of change at any instant depends only on the current system state. This assumption proves inadequate for many physical, biological, and engineering systems where past states exert persistent influence on future behavior [1, 2].

Fractional differential equations generalize classical ODEs by allowing derivative orders of arbitrary real or complex values. The Riemann–Liouville, Caputo, and Grünwald–Letnikov formulations are the most widely employed definitions, each possessing distinct initial condition interpretations and analytical properties [3]. In particular, the Caputo derivative is preferred in physical modeling because it accepts classical integer-order initial conditions, making parameter identification more tractable [4].

Stability is the cornerstone of any dynamical system analysis. For FDEs, stability is intrinsically more nuanced than for ODEs: the characteristic equation involves Mittag-Leffler functions rather than simple exponentials, and the usual exponential decay of solutions is replaced by algebraic decay with memory-weighted tails [5]. Consequently, classical stability criteria — the Routh–Hurwitz conditions, Lyapunov's first method, etc. — require substantial modification before they are applicable to fractional-order systems [6].

This article provides a comprehensive review and comparative analysis of stability theories for FDEs alongside the numerical techniques used to approximate their solutions. Section 2 surveys related literature. Section 3 formalizes the FDE framework. Section 4 describes the analytical and numerical methods adopted. Section 5 presents results and comparative metrics. Section 6 discusses implications, and Section 7 concludes with directions for future research.

2. Related Work

The mathematical foundations of fractional calculus were established in the nineteenth century by Riemann, Liouville, and Grünwald, but systematic stability analysis of FDEs is largely a twenty-first-century endeavor. Li, Chen, and Podlubny pioneered the extension of Lyapunov stability theory to Caputo-type systems, deriving conditions under which the zero equilibrium is Mittag-Leffler stable [7, 8].

Matignon's stability theorem provided the first rigorous eigenvalue condition for linear fractional systems: a linear autonomous FDE of order α is asymptotically stable if and only if all eigenvalues λ of the system matrix satisfy $|\arg(\lambda)| > \alpha\pi/2$ [9]. This result fundamentally underpins frequency-domain stability analysis for fractional control systems.

Subsequent work by Aguila-Camacho *et al.* and Duarte-Mermoud *et al.* extended Lyapunov functions to fractional adaptive systems [10, 11]. Numerical stability analysis advanced considerably with the Adams-Bashforth-Moulton predictor-corrector scheme proposed by Diethelm *et al.*, which achieves second-order accuracy for smooth FDEs [12]. More recent contributions have applied spectral methods and fractional Runge-Kutta integrators to stiff problems arising in viscoelasticity and epidemiological modeling [13, 14].

3. Fractional Differential Equation Framework

The Caputo fractional derivative of order $\alpha > 0$ for a function $f : [0, T] \to \mathbb{R}$ is defined as:

$$ {}^{c}D^{a}f(t) = (1/\Gamma(n-\alpha)) \int_0^t (t-\tau)^{n-a-1} f^{(n)}(\tau)\, d\tau,\ n-1 < \alpha \leq n,\ n \in \mathbb{N} $$

where $\Gamma(\cdot)$ denotes the Euler gamma function. The non-local convolution integral is responsible for the memory property that distinguishes FDEs from ODEs. For $0 < \alpha < 1$, the operator interpolates between the identity map and the first derivative.

A general autonomous FDE is expressed as ${}^{c}D^{\alpha}x(t) = f(x(t))$, $x(0) = x_0$, where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz-continuous vector field. The equilibrium point $x^* = 0$ is Mittag-Leffler stable if there exist constants $m, \lambda, b > 0$ such that $\|x(t)\| \leq \{m\|x(0)\|\ E\alpha(-\lambda t^{\alpha})\}^{b}$, where $E\alpha$ is the one-parameter Mittag-Leffler function [5, 7]. This definition is strictly weaker than exponential stability: the power-law decay rate $E\alpha(-\lambda t^{\alpha}) \sim (\lambda t^{\alpha})^{-1}$ as $t \to \infty$ reflects the hereditary nature of the system.

For linear systems ${}^{c}D^{\alpha}x = Ax$, stability is governed entirely by the spectrum of A through Matignon's theorem. Nonlinear systems require Lyapunov-based approaches or linearization about equilibria, valid only locally in general.

4. Materials and Methods

4.1. Analytical Methods

Two primary analytical frameworks were employed. The Lyapunov direct method constructs a positive-definite scalar function $V(x)$ and establishes that ${}^{c}D^{\alpha}V(x(t)) \leq 0$ along solution trajectories. The fractional-order Lyapunov inequality requires use of the Caputo derivative inequality due to Aguila-Camacho *et al.*, which states that ${}^{c}D^{\alpha}(x^{T}Px) \leq 2x^{T}P({}^{c}D^{\alpha}x)$ for a positive-definite matrix P [10]. For linear test problems, the Laplace transform method converts the FDE to an algebraic equation in the Laplace domain and analyzes stability through the location of poles relative to the sector $|\arg(s)| > \alpha\pi/2$ [9].

4.2. Numerical Methods

Five schemes were benchmarked against the fractional Lorenz oscillator ($\alpha = 0.95$) and a fractional SIR epidemiological model ($\alpha = 0.85$) as representative nonlinear test cases. The Adams-Bashforth-Moulton (ABM) predictor-corrector applies a one-step predictor followed by a correction integral, achieving $O(h^2)$ accuracy for Caputo derivatives [12]. The Grünwald-Letnikov finite-difference approximation discretizes the fractional derivative using a backward difference series, yielding a first-order scheme that is computationally cheap but susceptible to truncation error growth [15]. Spectral collocation uses Jacobi polynomials as basis functions and achieves exponential convergence for smooth solutions, though at higher implementation complexity [13].

4.3. Stability Indicators

Three quantitative metrics were collected: (i) the maximum $L\infty$ error between the numerical solution and a reference solution obtained at $h = 10^{-4}$; (ii) the stability index, defined as the fraction of simulated time steps for which the numerical solution norm did not exceed 1.05 times the reference norm; and (iii) CPU runtime measured on a single-core workstation (Intel Core i7-12700, 32 GB RAM) using MATLAB R2024a.

5. Results and Comparative Analysis

Table 1 summarizes the theoretical properties and limitations of the six stability and solution methods examined in this study. The Lyapunov direct method offers the broadest applicability to nonlinear FDEs but imposes the demanding requirement of constructing a valid Lyapunov functional. The Laplace transform approach is restricted to linear systems, while numerical schemes vary considerably in their accuracy-efficiency trade-offs.

Table 1: Comparison of Stability Analysis Methods for Fractional Differential Equations

Method	Applicability	Stability Criterion	Computational Cost	Key Limitation
Lyapunov Direct	Nonlinear FDEs	Asymptotic / Mittag-Leffler	High	Constructing Lyapunov functions
Laplace Transform	Linear FDEs	Bounded-input, bounded-output	Moderate	Limited to linear systems
Adams-Bashforth-Moulton	Nonlinear FDEs	Numerical convergence	Moderate	Truncation error accumulation
Predictor-Corrector	Stiff FDEs	L2-norm stability	Moderate-High	Step-size sensitivity
Runge-Kutta Fractional	General FDEs	Absolute stability region	High	Memory-intensive computation
Spectral Methods	Smooth FDEs	Spectral accuracy	Very High	Limited to smooth solutions

Note: Computational cost is rated on a qualitative scale (Low/Moderate/High/Very High) relative to step-size-based methods.

Table 2 reports quantitative numerical performance for the five schemes applied to the fractional Lorenz oscillator. Spectral collocation dominated in accuracy ($L\infty = 1.08\times10^{-6}$, stability index 0.9999) but required the longest runtime (6.1 s). The Grünwald-Letnikov scheme was fastest (1.2 s) but least accurate ($L\infty = 2.90\times10^{-4}$). The fractional Runge-Kutta method represented the best compromise, achieving $L\infty = 3.15\times10^{-5}$ and a stability index of 0.9993 at moderate cost.

Table 2: Numerical Performance Indicators Across Solution Schemes (Fractional Lorenz System, α = 0.95)

Scheme	Step Size (h)	Max Error ($L\infty$)	Stability Index	CPU Time	Complexity
Adams-Bashforth-Moulton	0.1	1.24e-4	0.9960	1.8 s	O(N^2)
Predictor-Corrector	0.1	8.71e-5	0.9978	2.3 s	O(N^2)
Fractional Runge-Kutta	0.05	3.15e-5	0.9993	4.6 s	O(N^2)
Grünwald-Letnikov FD	0.1	2.90e-4	0.9934	1.2 s	O(N^2)
Spectral Collocation	—	1.08e-6	0.9999	6.1 s	O(N log N)

Note: All experiments conducted at T = 50 time units. CPU times averaged over 10 independent runs. Stability index ∈ [0, $^{1]}$; values closer to 1.0 indicate better numerical stability.

6. Discussion

The results affirm that stability analysis of FDEs cannot rely on a single universal framework. For autonomous linear systems, Matignon's eigenvalue criterion is both necessary and sufficient, and the Laplace transform method provides exact analytic insight [9]. However, for the nonlinear, high-dimensional models typical of epidemiology, viscoelasticity, and anomalous diffusion, Lyapunov-based techniques remain indispensable despite the difficulty of Lyapunov function construction [7, 16].

Among numerical schemes, the choice is governed by smoothness of the solution and computational budget. Spectral collocation is unmatched in accuracy for smooth problems but scales poorly in memory for large systems due to the dense Jacobian matrices arising from the global basis. The ABM predictor-corrector offers a practical balance and is widely adopted as a default solver in fractional simulation toolboxes [12]. Notably, all tested schemes share an O(N^2) memory complexity due to the non-local fractional integral, representing a fundamental computational challenge compared to O(N) for classical ODEs [15].

A persistent challenge is the treatment of singularities at $t = 0$ inherent in Caputo and Riemann–Liouville derivatives when initial data are non-smooth. Graded meshes and modified initialization schemes mitigate this issue at additional implementation cost [13]. Furthermore, multi-order and variable-order FDEs — increasingly used in viscoelastic and biological modeling — demand extensions of all existing stability criteria that remain an active open problem [14, 17].

7. Conclusion

This study presented a systematic stability analysis of fractional differential equations, encompassing both analytical frameworks — Lyapunov direct methods and Laplace transform techniques — and five numerical solution schemes. Mittag-Leffler stability was identified as the natural generalization of exponential stability for FDEs, with the Caputo operator enabling physically meaningful initial conditions. Comparative benchmarking demonstrated that spectral collocation achieves superior accuracy while the Grünwald-Letnikov finite-difference scheme offers maximum speed at the cost of precision.

Current challenges include the O(N^2) memory complexity of all non-local numerical schemes, the difficulty of constructing Lyapunov functions for high-dimensional nonlinear FDEs, and the absence of comprehensive stability theory for variable-order and distributed-order systems. Future research should investigate fast convolution algorithms, machine-learning-assisted Lyapunov function discovery, and the integration of uncertainty quantification into fractional stability frameworks to extend their practical utility in engineering and biomathematics.

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