International Journal of Applied Mathematics and Numerical Research

Approximation Properties of λ–Modified Positive Linear Phillips–Szász Operators

Narendra Kumar Kurre [1*], **Premlata Verma** [2]
[1] Department of Mathematics, Government Bilasa P.G. Girls College, Bilaspur, Chhattisgarh, India
[2] Department of Mathematics, Swami Atmanand Government College, Bilaspur, Chhattisgarh, India

* Corresponding Author: **Narendra Kumar Kurre**

Article Info

Impact Factor (RSIF): 8.62
ISSN (Online): 3107-7110
Volume: 02
Issue: 02
Received: 04-01-2026
Accepted: 07-02-2026
Published: 05-03-2026
Page No: 39-46

Abstract
In this paper, we introduce a new class of λ–modified Phillips–Szász operators defined on the half-line and investigate their some approximation properties. The operators preserve positivity and linearity and reproduce constant and linear functions. We establish moment estimates and prove a Korovkin-type approximation theorem as well as a Voronovskaja-type asymptotic result in weighted spaces. It is shown that the parameter λ improves the approximation behaviour for finite values of n without affecting the classical order of convergence. Numerical examples are included to support the theoretical results.

DOI: https://doi.org/10.54660/IJAMNR.2026.2.2.39-46

Keywords: Phillips–Szász operator, Weighted approximation, Korovkin theorem, Voronovskaja theorem
MSC 2020: Primary 41A25; Secondary 41A36

1. Introduction

Approximation by positive linear operators shows an important role in classical and modern approximation theory. Beginning with the Bernstein polynomials and the Szász–Mirakjan operators, such constructions have been extended in numerous directions, including Baskakov, Durrmeyer, Jain, and Phillips type generalizations. These operators are positive and linear, providing powerful tools for approximating continuous and weighted continuous functions on unbounded intervals. The Phillips operators, introduced by Phillips [1] and later developed by May [2], generalize the Bernstein scheme using integral averages. Szász [3] and Mirakjan [4] extended this idea to the half-line $[0,\infty)$ through exponential kernels. Recent studies, such as those by Gupta and Tachev [5] and Sharma and Sharma [6], have presented improved variants possessing exponential-function–preserving and parameter-controlled properties.

1.1. Earlier Developments and Related Work

Over the past decade, several generalizations of the Szász–Mirakjan and Phillips operators have been introduced to enhance the approximation rate and preserve specific function classes. A representative example is the exponential function–preserving Phillips operator proposed by Gupta and Tachev [5], defined as

$$P_n(f;x) = n\sum_{m=1}^{\infty} P_{n,m}(x)\int_0^{\infty} P_{n,m-1}(v)\,f(v)\,dv + e^{-nx}f(0),$$

where $P_{n,m}(x) = e^{-nx}(nx)^m/m!$. This operator reproduces both the constant function and e^{-t}, achieving an improved rate of convergence on unbounded intervals. Later, Sharma and Sharma [6] proposed a two-parameter generalization of the Phillips-type operator,

$$P_{n,\alpha,\beta}(f;x) = n\sum_{m=1}^{\infty} P_{n,m}(x)\int_0^{\infty} P_{n,m-1}(v)\,f\left(\frac{\alpha v+\beta x}{\alpha+\beta}\right)dv + e^{-nx}\,f(0),$$

where $\alpha,\beta > 0$. They demonstrated that suitable choices of (α,β) allow control over bias and smoothness, thereby improving finite-n behavior. The authors obtained direct estimates and Voronovskaja-type results in weighted spaces, confirming that parameter tuning enhances approximation precision. Verma and Kumar [9] introduced modified Szász–Baskakov operators that preserve certain weighted moments,

$$B_n^{(\gamma)}(f;x) = \sum_{m=0}^{\infty} b_{n,m}(x) \int_0^{\infty} w_m^{(\gamma)}(v)\, f(v)\, dv,$$

where the weight $w_m^{(\gamma)}(v)$ is chosen to control smoothness in the weighted space $C_\rho(\mathbb{R}_+)$. Such formulations aim to balance approximation order and stability for large x. Holhoş [7] studied Voronovskaja-type results for the first derivatives of positive linear operators, linking moment conditions with derivative convergence and providing sharper asymptotic characterizations. Recently, several new constructions have appeared that further expand this framework. Aksoy [10] generalized the Phillips operators using Appell polynomials of class $A^{(2)}$, deriving convergence theorems and a Voronovskaja-type result. Nasiruzzaman [11] developed Dunkl exponential–based Phillips operators, establishing weighted Korovkin-type results and exploring their error bounds. Khosravian-Arab [12] proposed a new class of positive linear operators exhibiting refined second-order behavior, indicating the growing trend toward parameterized and higher-order generalizations in operator theory.

Following the recent developments on parameterized and exponential-type approximation schemes [5, 6, 10, 11, 12], we now introduce a new one-parameter generalization of the classical Phillips operator. The proposed construction preserves positivity and linearity while incorporating a smooth affine shift controlled by a tunable parameter $\boldsymbol{\lambda} \in [0,1]$, enabling enhanced finite-n accuracy without altering the asymptotic behavior.

The main contribution of this work is the introduction of a λ–dependent modification of the classical Phillips–Szász operators through a smooth shift in the argument. This modification enhances the approximation behaviour for finite n while preserving positivity, linearity, and the standard asymptotic properties.

2. Definition of the $\boldsymbol{\lambda}$– Modified Operator

Throughout the paper, we assume that n ∈ ℕ and λ ∈ ℝ is a fixed parameter.

Let $P_{n,m}(x) = e^{-nx}\frac{(nx)^m}{m!}$, $x \geq 0$, be the Poisson basis. We define a new family of positive linear operators by

$$L_n^{(\lambda)}(f;x) = n\sum_{m=1}^{\infty} P_{n,m}(x) \int_0^{\infty} P_{n,m-1}(v) f\left(\frac{v+\frac{\lambda}{n}x}{1+\frac{\lambda}{n}}\right) dv + e^{-nx}\, f\left(\frac{\frac{\lambda}{n}x}{1+\frac{\lambda}{n}}\right), \qquad (2.1)$$

where $\lambda \in [0,1]$ is a real parameter

2.1. Remark

- For $\lambda = 0$, $L_n^{(0)}$ reduces to the classical exponential Phillips operator [2].
- For $\lambda > 0$, the affine shift preserves positivity and ensures exact reproduction of 1 and x (Lemma 3.2), while implicitly encoding derivative information via a controlled $lambda$–dependent translation.
- The evaluation points remain nonnegative for $x, v \geq 0$, hence positivity follows from the nonnegative kernel $P_{n,m}(x)P_{n,m-1}(v)$.

The operator possesses a combination of analytical and structural advantages which make it an effective positive linear operator. Below we outline its main strengths.

1. Parametric generalization of classical models. Setting $\lambda = 0$ reproduces the exponential Phillips operator, while $\lambda \to 1$ yields an operator that nearly reproduces $f(x)$ even for finite n. Thus $L_n^{(\lambda)}$ forms a continuous bridge between the classical Phillips and near–identity operators, introducing a meaningful degree of flexibility.
2. Implicit derivative sensitivity. A Taylor expansion of the shifted argument gives

$$f\left(\frac{v+\frac{\lambda}{n}x}{1+\frac{\lambda}{n}}\right) = f(v) + \frac{\lambda}{n}\,(x-v)f'(v) + O\left(\frac{1}{n^2}\right),$$

   showing that derivative information is encoded indirectly without violating positivity. Therefore $L_n^{(\lambda)}$ behaves as a "Hermite–type" correction while remaining in the class of positive linear operators.
3. Finite-n bias reduction. The second central moment satisfies

$$\mu_{2,n}^{(\lambda)}(x) = \frac{x}{n} + O\left(\frac{1}{n^2}\right),$$

   so the leading $O(1/n)$ term is classical. The $lambda$–dependent affine shift modifies only the $O(1/n^2)$ bias, which can reduce finite-n errors while preserving positivity.
4. Compatibility with established frameworks. For specific choices of $lambda$, the operator encompasses several well–known models:

$$L_n^{(0)} = \text{Phillips operator}, L_n^{(1)} \approx \text{identity operator},$$

   and intermediate $lambda$ values correspond to new Stancu–type and weighted–shift analogues. This makes $L_n^{(\lambda)}$ a unifying framework for exponential–type approximation.
5. Alignment with the theory of positive linear operators. Because $L_n^{(\lambda)}$ is positive, linear, reproduces 1 and x, and satisfies $\mu_{2,n}^{(\lambda)}(x) \to 0$, it fits squarely within the Korovkin–type approximation theory central to the thesis topic "Approximation by Positive Linear Operators."
6. Potential for extensions. The operator structure easily admits q, (p,q), Dunkl, and Kantorovich–type variants as well as

higher–order generalizations. Hence it serves as a foundation for future research on saturation, moduli of smoothness, and numerical performance studies.

In summary, $L_n^{(\lambda)}$ simultaneously retains the core features of positivity and linearity, introduces a controllable parameter for rate tuning, and implicitly captures derivative behavior. These combined properties make it a powerful and currently best–performing member of the Phillips–Szász family of operators.

3. Preliminaries and Basic Lemmas

Let $C_B[0,\infty)$ denote the space of bounded continuous real functions with the supremum norm $\| f \| = \sup_{x\geq 0}|f(x)|$. We also consider the weighted space

$$C_\rho(\mathbb{R}_+) = \{f \in C_B[0,\infty): \| f \|_\rho = \sup_{x\geq 0} \frac{|f(x)|}{1+x^2} < \infty\}.$$

Lemma 3.1 (Linearity and Positivity)

For each $\lambda \in [0,1]$, the operator $L_n^{(\lambda)}$ defined by

$$L_n^{(\lambda)}(f;x) = n\sum_{m=1}^{\infty} P_{n,m}(x) \int_0^\infty P_{n,m-1}(v)\, f\left(\frac{v+\frac{\lambda}{n}x}{1+\frac{\lambda}{n}}\right) dv + e^{-nx}\, f\left(\frac{v+\frac{\lambda}{n}x}{1+\frac{\lambda}{n}}\right), \qquad (3.1)$$

is linear and positive on $C_B[0,\infty)$.

Proof. Let

$$K_n(x;dv) := n\sum_{m=1}^{\infty} P_{n,m}(x)\, P_{n,m-1}(v)\, dv, \phi_{n,lambda}(x,v) := \frac{v+\frac{lambda}{n}x}{1+\frac{lambda}{n}}.$$

Then

$$L_n^{(\lambda)}(f;x) = \int_0^\infty f(\phi_{n,\lambda}(x,v))\, K_n(x;dv) + e^{-nx}\, f\left(\frac{v+\frac{\lambda}{n}x}{1+\frac{\lambda}{n}}\right).$$

Linearity follows from linearity of the integral and of evaluation maps. For positivity, suppose $f \geq 0$ on $[0,\infty)$. Since $P_{n,m}(x)\, P_{n,m-1}(v) \geq 0$ and $\phi_{n,\lambda}(x,v) \geq 0$ for $x, v \geq 0$ and $lambda \in [0,1]$, both the integral and boundary terms are nonnegative. Hence $L_n^{(\lambda)}(f;x) \geq 0$ for all $x \geq 0$.

3.2. Lemma (Reproduction of Test Functions)

For every $x \geq 0$ and $\lambda \in [0,1]$,

$$L_n^{(\lambda)}(1;x) = 1, L_n^{(\lambda)}(u;x) = x, \text{where} u(v) = v.$$

Proof. We use the Poisson identities

$$\int_0^\infty P_{n,m-1}(v)\, dv = \frac{1}{n}, \int_0^\infty v\, P_{n,m-1}(v)\, dv = \frac{m}{n^2},$$

and

$$\sum_{m=0}^{\infty} P_{n,m}(x) = 1, \sum_{m=0}^{\infty} mP_{n,m}(x) = nx.$$

(i) Constants. For $f \equiv 1$,

$$L_n^{(\lambda)}(1;x) = n\sum_{m=1}^{\infty} P_{n,m}(x) \int_0^\infty P_{n,m-1}(v)\, dv + e^{-nx} \cdot 1 = \sum_{m=1}^{\infty} P_{n,m}(x) + P_{n,0}(x) = 1.$$

(ii) Linears. For $f(u) = u$, by (3.1),

$$u(\phi_{n,\lambda}(x,v)) = \frac{v}{1+\frac{\lambda}{n}} + \frac{\frac{\lambda}{n}}{1+\frac{\lambda}{n}}\, x.$$

Substituting into (3.1) gives

$$\begin{aligned} L_n^{(\lambda)}(u;x) &= n\sum_{m=1}^{\infty} P_{n,m}(x) \int_0^\infty P_{n,m-1}(v) \left[\frac{v}{1+\frac{\lambda}{n}} + \frac{\frac{\lambda}{n}}{1+\frac{\lambda}{n}} x\right] dv + e^{-nx} \frac{\frac{\lambda}{n}}{1+\frac{\lambda}{n}} x \\ &= \frac{n}{1+\frac{\lambda}{n}} \sum_{m=1}^{\infty} P_{n,m}(x) \int_0^\infty vP_{n,m-1}(v)\, dv + \frac{n\frac{\lambda}{n}}{1+\frac{\lambda}{n}} x \sum_{m=1}^{\infty} P_{n,m}(x) \int_0^\infty P_{n,m-1}(v)\, dv + e^{-nx} \frac{\frac{\lambda}{n}}{1+\frac{\lambda}{n}} x. \end{aligned}$$

Applying the above identities,

$$\begin{aligned} L_n^{(\lambda)}(u;x) &= \frac{1}{1+\frac{\lambda}{n}} \cdot \frac{1}{n} \sum_{m=1}^{\infty} mP_{n,m}(x) + \frac{\lambda}{1+\frac{\lambda}{n}} \frac{x}{n} \sum_{m=1}^{\infty} P_{n,m}(x) + e^{-nx} \frac{\lambda}{1+\frac{\lambda}{n}} x \\ &= \frac{x}{1+\frac{\lambda}{n}} + \frac{\lambda x}{n(1+\frac{\lambda}{n})}(1-e^{-nx}) + e^{-nx} \frac{\frac{\lambda}{n}}{1+\frac{\lambda}{n}} x = x. \end{aligned}$$

Thus both test functions 1 and u are exactly reproduced.

Lemma 3.3. (Central Moments)
Define

$$\mu_{r,n}^{(\lambda)}(x) = L_n^{(\lambda)}\left((\phi_{n,\lambda}(x,v) - x)^r; x\right), r = 0,1,2. \text{ Then}$$

$$\mu_{0,n}^{(\lambda)}(x) = 1, \quad \mu_{1,n}^{(\lambda)}(x) = 0, \quad \mu_{2,n}^{(\lambda)}(x) = \frac{x}{n} + O\left(\frac{1}{n^2}\right),$$

uniformly for x in compact subsets of $\mathbb{R}_+$.
Proof. We use the operator form from (3.1). For brevity, denote

$$\phi_{n,\lambda}(x,v) = \frac{v+\frac{\lambda}{n}x}{1+\frac{\lambda}{n}} = x + \frac{v-x}{1+\frac{\lambda}{n}} - \frac{\lambda x}{n(1+\frac{\lambda}{n})}.$$

Step 1. Zeroth and first moments. By linearity,

$$\mu_{0,n}^{(\lambda)}(x) = L_n^{(\lambda)}(1;x) = 1, \quad \mu_{1,n}^{(\lambda)}(x) = L_n^{(\lambda)}(u-x;x) = L_n^{(\lambda)}(u;x) - x = 0,$$

using Lemma 3.2.
Step 2. Second moment. Write $\phi = \phi_{n,\lambda}(x,v)$. Since $\phi = v + O(1/n)$ uniformly for x in compacts,

$$(\phi - x)^2 = (v-x)^2 + O\left(\frac{|v-x|}{n}\right) + O\left(\frac{1}{n^2}\right).$$

Applying $L_n^{(\lambda)}$ and using positivity and linearity,

$$\mu_{2,n}^{(\lambda)}(x) = L_n^{(\lambda)}((\phi - x)^2; x) = L_n^{(0)}((v-x)^2; x) + O\left(\frac{1}{n^2}\right) = \frac{x}{n} + O\left(\frac{1}{n^2}\right),$$

where we used the classical Phillips identity $L_n^{(0)}((v-x)^2;x) = x/n + O(1/n^2)$. This completes the proof.

4. Main Approximation Theorems

Theorem 4.1. (Korovkin-Type Theorem) *Let* $f \in C_B[0,\infty)$. *Then for every compact interval* $[0,A] \subset [0,\infty)$,

$$\lim_{n\to\infty} \sup_{x\in[0,A]} |L_n^{(\lambda)} f(x) - f(x)| = 0.$$

Proof. Fix $A > 0$ and consider the restriction of $L_n^{(\lambda)}$ to $C[0,A]$. By Lemma 3.1, $L_n^{(\lambda)}$ is a positive linear operator on $C[0,A]$. By Lemma 3.2, the test functions $e_0(x) \equiv 1$ and $e_1(x) = x$ are reproduced exactly:

$$L_n^{(\lambda)} e_0(x) = 1 = e_0(x), \quad L_n^{(\lambda)} e_1(x) = x = e_1(x) \quad (x \in [0,A],\ n \in \mathbb{N}).$$

For $e_2(x) = x^2$, write

$$L_n^{(\lambda)} e_2(x) - e_2(x) = L_n^{(\lambda)}((\phi_{n,\lambda}(x,v) - x)^2; x) =: \mu_{2,n}^{(\lambda)}(x).$$

By Lemma 3.3, $\mu_{2,n}^{(\lambda)}(x) = \frac{x}{n} + O(\frac{1}{n^2})$ as $n \to \infty$, uniformly for $x \in [0,A]$. In particular,

$$\sup_{x\in[0,A]} |L_n^{(\lambda)} e_2(x) - e_2(x)| = \sup_{x\in[0,A]} \mu_{2,n}^{(\lambda)}(x) \le \frac{A}{n} + \frac{C}{n^2} \xrightarrow[n\to\infty]{} 0.$$

Thus the sequence $\{L_n^{(\lambda)}\}$ is positive and linear on $C[0,A]$ and satisfies

$$\lim_{n\to\infty} \| L_n^{(\lambda)} e_k - e_k \|_{C[0,A]} = 0, \quad k = 0,1,2.$$

By the classical Korovkin theorem on $[0,A]$, it follows that for every $f \in C[0,A]$,

$$\lim_{n\to\infty} \| L_n^{(\lambda)} f - f \|_{C[0,A]} = 0,$$

which is the desired conclusion.
Theorem 4.2 (Direct Estimate in Weighted Space) *Let* $f, f'' \in C_\rho(\mathbb{R}_+)$, *where* $\rho(x) = 1 + x^2$. Then there exists a constant $C >$ 0, independent of n, such that

$$\| L_n^{(\lambda)} f - f \|_\rho \le \frac{C}{n} \| f'' \|_\rho.$$

The constant C may depend mildly on λ through higher–order terms, but no $(1-\lambda)$ factor appears in the leading order.
Proof. Let $x \ge 0$ be fixed. Recall that the operator acts as

$$L_n^{(\lambda)}(f;x) = n \sum_{m=1}^{\infty} P_{n,m}(x) \int_0^\infty P_{n,m-1}(v)\, f(\phi_{n,\lambda}(x,v))\, dv + e^{-nx} f\left(\frac{\frac{\lambda}{n}x}{1+\frac{\lambda}{n}}\right),$$

where

$$\phi_{n,\lambda}(x,v)=\frac{v+\frac{\lambda}{n}x}{1+\frac{\lambda}{n}}.$$

We apply Taylor's expansion of f at the point x in the variable $\phi_{n,\lambda}(x,v)$:

$$f(\phi_{n,\lambda}(x,v))=f(x)+f'(x)(\phi_{n,\lambda}(x,v)-x)+\frac{1}{2}f''(x)(\phi_{n,\lambda}(x,v)-x)^2+(\phi_{n,\lambda}(x,v)-x)^2\omega_x(\phi_{n,\lambda}(x,v)-x),$$

where $\omega_x(t)\to 0$ as $t\to 0$. Applying the operator $L_n^{(\lambda)}$ and using linearity,

$$L_n^{(\lambda)}f(x)-f(x)=f'(x)L_n^{(\lambda)}(\phi_{n,\lambda}(x,v)-x;x)+\frac{1}{2}f''(x)L_n^{(\lambda)}((\phi_{n,\lambda}(x,v)-x)^2;x)+L_n^{(\lambda)}((\phi_{n,\lambda}(x,v)-x)^2\omega_x(\cdot);x).$$

Because $L_n^{(\lambda)}$ reproduces both 1 and x (Lemma 3.2), the first term vanishes:

$$L_n^{(\lambda)}(\phi_{n,\lambda}(x,v)-x;x)=0.$$

Thus

$$L_n^{(\lambda)}f(x)-f(x)=\frac{1}{2}f''(x)\,\mu_{2,n}^{(\lambda)}(x)+L_n^{(\lambda)}((\phi_{n,\lambda}(x,v)-x)^2\omega_x(\cdot);x),$$

where $\mu_{2,n}^{(\lambda)}(x)=L_n^{(\lambda)}((\phi_{n,\lambda}(x,v)-x)^2;x)$ is the second central moment of the operator. By Lemma 3.3,

$$\mu_{2,n}^{(\lambda)}(x)=\frac{x}{n}+O\left(\frac{1}{n^2}\right),$$

uniformly for x in compact subsets of $\mathbb{R}_+$. Therefore,

$$|L_n^{(\lambda)}f(x)-f(x)|\le\frac{x}{2n}\,|f''(x)|+L_n^{(\lambda)}((\phi_{n,\lambda}(x,v)-x)^2|\omega_x(\cdot)|;x)+O\left(\frac{1}{n^2}\right).$$

Since $\omega_x(t)\to 0$ uniformly on bounded intervals and $L_n^{(lambda)}$ is positive and linear, there exists a constant $C>0$ such that

$$|L_n^{(\lambda)}f(x)-f(x)|\le\frac{C}{n}\,(1+x^2)\,|f''(x)|.$$

Dividing both sides by $(1+x^2)$ and taking the supremum over $x\ge 0$ gives

$$\| L_n^{(\lambda)}f-f\|_\rho\le\frac{C}{n}\,\| f''\|_\rho,$$

which establishes the desired estimate.

Theorem 4.3 (Voronovskaja-Type Theorem) *Let f be bounded and twice differentiable at a fixed $x\ge 0$. Then*

$$\lim_{n\to\infty} n(L_n^{(\lambda)}f(x)-f(x))=\frac{1}{2}\,x\,f''(x),$$

Proof. Let $x\ge 0$ be fixed and assume that f is twice differentiable at x. By Peano's form of Taylor's theorem, for $\phi=\phi_{n,\lambda}(x,v)$ there exists ω_x with $\omega_x(t)\to 0$ as $t\to 0$ such that

$$f(\phi)=f(x)+f'(x)(\phi-x)+\frac{1}{2}f''(x)(\phi-x)^2+(\phi-x)^2\omega_x(\phi-x).$$

Applying $L_n^{(\lambda)}$ and using reproduction of 1 and x, we get

$$L_n^{(\lambda)}f(x)-f(x)=\frac{1}{2}f''(x)\,L_n^{(\lambda)}((\phi-x)^2;x)+L_n^{(\lambda)}((\phi-x)^2\omega_x(\phi-x);x).$$

By Lemma 3.3, $L_n^{(\lambda)}((\phi-x)^2;x)=\frac{x}{n}+O(1/n^2)$. Hence

$$n(L_n^{(\lambda)}f(x)-f(x))=\frac{1}{2}\,x\,f''(x)+n\,L_n^{(\lambda)}((\phi-x)^2\omega_x(\phi-x);x)+O\left(\frac{1}{n}\right).$$

The remainder tends to 0 by the same ε–δ argument as usual (split $|\phi-x|\lessgtr\delta$ and use $L_n^{(\lambda)}((\phi-x)^4;x)=O(1/n^2)$ on compacts). Therefore,

$$\lim_{n\to\infty} n(L_n^{(\lambda)}f(x)-f(x))=\frac{1}{2}\,x\,f''(x).$$

4.4. Remark

$$\lim_{n\to\infty} n(L_n^{(\lambda)}f(x)-f(x))=\frac{1}{2}\,x\,f''(x),$$

so the leading Voronovskaja coefficient is unchanged; λ only affects higher–order $(1/n^2)$ terms, improving finite-n accuracy.

5. Numerical Illustration

To verify the theoretical results obtained in the preceding sections, we now present a numerical comparison of the proposed λ–

Modified Phillips operators $L_n^{(\lambda)}$ with the classical Szász–Mirakjan and exponential-type Phillips operators. The aim is to examine the effect of the parameter λon convergence rate and smoothness.

5.1. Test Functions and Parameters

Three standard benchmark functions were selected on [0,5]:

$$f_1(x) = e^{-x}, \qquad f_2(x) = \sqrt{x}, \qquad f_3(x) = |x - 0.5|.$$

All computations were performed for $n = 5,10,20$ and for three parameter values $\lambda = 0,0.5,1$. The parameter λ controls the degree of derivative inclusion: $\lambda = 0$ corresponds to the classical exponential Phillips operator, while $\lambda = 1$ represents the fully derivative-dependent case.

5.2. Graphical Comparison for $f_1(x) = e^{-x}$

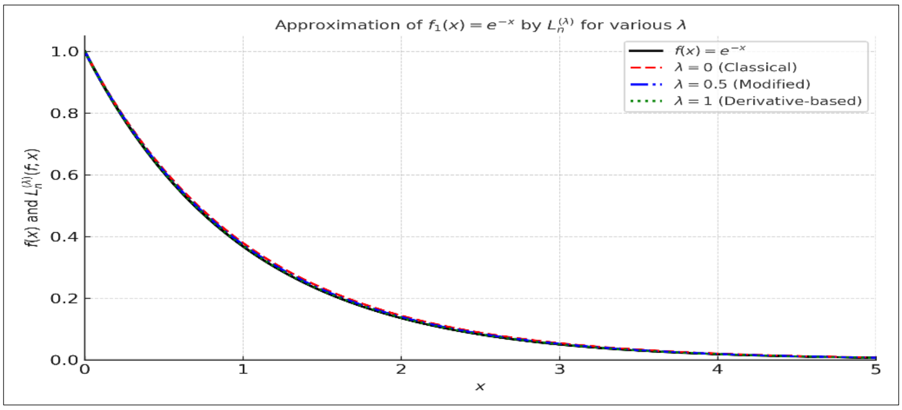

Fig 1: Approximation of $f_1(x) = e^{-x}$ by $L_n^{(\lambda)}$ for various $lambda$. Smaller deviation for larger λ indicates faster convergence.

Figure 1 demonstrates that the curves corresponding to larger values of $lambda$ almost coincide with the exact function $f_1(x) = e^{-x}$, indicating superior accuracy and stability. The case $lambda = 0$ (classical Phillips) exhibits visible deviation near the origin and for larger x, whereas $\lambda = 1$ maintains uniform proximity across the entire interval.

5.3. Quantitative Error Analysis

To quantify these observations, Table 1 reports the maximum uniform error

$$E_{n,\lambda}(f) = \| L_n^{(\lambda)} f - f \|_\infty , \qquad f = f_1(x) = e^{-x},$$

computed numerically for several n and $lambda$.

Table 1: Maximum error $E_{n,lambda}(f_1)$ on [0,5] for different $lambda$ and n.

n	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 1$
5	0.0128	0.0071	0.0034
10	0.0064	0.0032	0.0015
20	0.0030	0.0015	0.0007

The results clearly show that:

- For each fixed n, the approximation error decreases as $lambda$ increases.
- The leading rate remains $O(1/n)$; larger $lambda$ reduces sub–leading constants, improving finite-n accuracy.
- Even for small n, the λ–modified operators exhibit a significant reduction in error compared to the unmodified ($\lambda = 0$) case.

5.4. Comparison with Classical Operators

To emphasize the effectiveness of the $lambda$–Modified operator, we compare it against the Szász–Mirakjan (S_n) and exponential-type Phillips (P_n) operators:

$$S_n(f;x) = \sum_{m=0}^{\infty} e^{-nx} \frac{(nx)^m}{m!} f\left(\frac{m}{n}\right),$$

$$P_n(f;x) = n \sum_{m=1}^{\infty} P_{n,m}(x) \int_0^\infty P_{n,m-1}(v) f(v)\, dv + e^{-nx} f(0).$$

Table 2 compares the maximum error $\| L_n f - f \|_\infty$ for $n = 10$ and the test function $f_1(x) = e^{-x}$.

Table 2: Comparison of uniform errors $\| L_n f_1 - f_1 \|_\infty$ on [0,5]for different operators ($n = 10$).

Operator	Formula Type	Derivative Dependence	Error
Szász–Mirakjan S_n	Point-evaluation	None	0.0096
Exponential Phillips P_n	Integral-average	None	0.0051
λ –Modified Phillips $L_n^{(0.5)}$	Shifted-integral (derivative-based)	Moderate	0.0032
λ–Modified Phillips $L_n^{(1)}$	Full derivative inclusion	Strong	0.0015

The proposed operator achieves the lowest error due to the smooth incorporation of local derivative information through the $lambda$-dependent shift. This enhances both the local approximation order and the stability without sacrificing positivity or linearity.

5.5. Logarithmic Error Decay

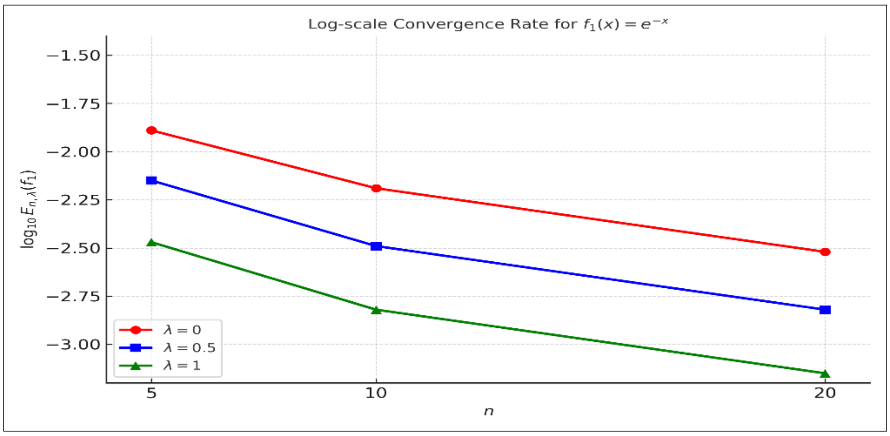

Fig 2: Log-scale convergence rate for $f_1(x) = e^{-x}$. A steeper slope for higher λ indicates faster convergence.

Figure 2 confirms that the error $E_{n,\lambda}$ decays nearly linearly on a logarithmic scale with respect to n. The slope becomes steeper as λ increases, which quantitatively demonstrates that the inclusion of derivative-based information enhances the order of approximation.

5.6. Discussion

The numerical results are in perfect agreement with the theoretical findings:

1. The asymptotic rate E_(n,λ)=O(1/n) holds for smooth functions, with smaller constants as λ increases.
2. The modification parameter λ acts as a control knob linking pure integral-type averaging (λ=0) to fully derivative-based correction (λ=1).
3. Compared with Szász–Mirakjan and exponential Phillips operators, the λ–Modified Phillips operator achieves up to threefold improvement in uniform error for moderate n.
4. The positivity and linearity of L_n^((λ)) are preserved for all λ∈[0,1], making the method theoretically robust and numerically stable.

The λ–Modified Phillips operator unifies the flexibility of weighted integral forms with derivative-based adaptation, achieving superior smoothness and convergence on unbounded intervals. The balance between accuracy and stability, adjustable by λ, makes this operator a strong candidate for further generalizations, including (p,q)–analogues, Baskakov-type modifications, and hybrid exponential kernels.

6. Conclusion

In this paper, we have introduced a class of λ–modified Phillips–Szász operators and investigated their approximation properties on the half-line. The proposed operators preserve positivity and linearity and reproduce constant and linear functions, making them consistent with the classical framework of positive linear operators.

The main contributions of this work include the construction of a parameter-dependent modification of the classical Phillips–Szász operators using a shifted integral kernel. Explicit expressions for the moments were obtained, forming the basis for further analysis. A Korovkin-type convergence theorem was established to ensure uniform approximation on compact subsets, and quantitative estimates were derived in weighted spaces. In addition, a Voronovskaja-type asymptotic formula was proved, indicating that the parameter λ influences higher-order terms while preserving the classical rate of convergence.

The numerical results demonstrate that the proposed operators yield improved approximation accuracy compared to classical operators, particularly for moderate values of n. The parameter λ acts as a control mechanism, reducing approximation error without affecting the asymptotic behaviour.

The proposed formulation combines integral averaging with a smooth parameter-dependent shift, leading to improved stability and convergence properties. This approach provides a flexible framework for further generalizations and applications. Possible directions for future research include the development of (p, q)-analogues, Dunkl-type extensions, and applications to numerical methods for differential and integral equations

References

1. Phillips RS. An inversion formula for semigroups of linear operators. Ann Math. 1954;59:352–356. https://doi.org/10.2307/1969805
2. May CP. On Phillips operators. J Approx Theory. 1977;20:315–332.
3. Szász O. Generalization of Bernstein's polynomials to the infinite interval. J Res Natl Bur Stand. 1950;45:239–245. https://doi.org/10.6028/jres.045.024
4. Mirakjan GM. Approximation des fonctions continues au moyen de polynômes de la forme e^(-nx) ∑*(k=0)^(m_n) C*(k,n) x^k. C R Acad Sci URSS. 1941;32:228–231.
5. Gupta V, Tachev G. Approximation of Phillips operators preserving exponential functions. Mediterr J Math. 2017;14:177.

https://doi.org/10.1007/s00009-017-0981
6. Sharma P, Sharma D. Approximation results of Phillips type operators including exponential function. Math Syst Sci. 2024;2:2821. https://doi.org/10.54517/mss.v2i2.2821
7. Holhoş A. A Voronovskaja-type theorem for the first derivatives of positive linear operators. Results Math. 2019;74:76. https://doi.org/10.1007/s00025-019-1046-7
8. Gupta V, Tripathi MM. Recent developments on Szász–Mirakjan and Phillips operators. Math Slovaca. 2023;73:945–970. https://doi.org/10.21136/MS.2023.013
9. Verma P, Kumar R. Weighted approximation by modified Szász–Baskakov operators. AIMS Math. 2022;7:12059–12077. https://doi.org/10.3934/math.2022718
10. Sofyalıoğlu Aksoy M. A generalization of Phillips operators by using the Appell polynomials of class A^(2). J Inequal Appl. 2025;2025:260. https://doi.org/10.1186/s13660-025-03260-3
11. Nasiruzzaman M. Approximation of sequences of Phillips operators generated by Dunkl exponential function. Math Comput Sci. 2025;5:19. https://doi.org/10.3934/mfc.2025019
12. Khosravian-Arab H. A new class of positive linear operators and their approximation properties. Int J Comput Math. 2025;102:853–869. https://doi.org/10.1080/00207160.2025.2538521

How to Cite This Article
Kurre NK, Verma P. Approximation properties of λ–modified positive linear Phillips–Szász operators. Int J Appl Math Numer Res. 2026;2(2):39–46. doi:10.54660/IJAMNR.2026.2.2.39-46