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On the Fixed Point and the Fixed Set Property of Three Newly Constructed Infinite Matrices and their Stability and a Topological Property of their Convergence Domain

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Abstract
Several researchers had worked on the fixed point of some iteration schemes. For instance, in Mann constructed an iteration scheme in a convex Banach space E that converged to the fixed-point of the transformation T defined on E while in, Dotson applied the iterative process introduced by Mann in 1953 to the approximation of fixed-point of quasi non-expansive mapping in Hilbert space and in uniformly convex and strictly convex Banach spaces. Furthermore, Rhoades in used an infinite matrix of weighted mean to determine the convergence of an iteration scheme to the fixed-point of the self-mapping defined on the iteration scheme in Banach space. This paper presents the fixed-point properties of three infinite matrices acting on some sequence spaces by classifying them to be either a source or sink fixed points. In addition, the stability of these fixed points are rigorously analysed, with each point being identified as either stable or unstable. Furthermore, we determine the fixed set or Fix(A) of the convergent domains of the operators in the sequence spaces, as well as their fixed subsets alongside with the conditions for stability. A significant topological property of these domains is established, underscoring the structural interplay between infinite matrix transformations and related fixed-points.

Keywords: Infinite Matrices, Fixed point, operator, stability, sink, source, isomorphic
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1. Introduction

The theory of infinite matrices occupies a central role in functional analysis, particularly in the study of sequence spaces and summability methods. Classically, infinite matrices arise as natural representations of linear operators between sequence spaces, and their properties such as boundedness, regularity, and conservativeness. The concept of infinite matrix $A = (a_{nk})$ which defines a linear transformation from one sequence space into another sequence space lies at the heart of many summability methods. If the new sequence say $\{t_n\}$, where $t_n = \sum_{k=1}^{\infty} a_{nk} x_k$ converges, then, the original sequence or series is said to be summable by the method $A = (a_{nk})$. These infinite matrices transform some divergent sequences and series into convergent ones under some conditions called regularity conditions given by Silverman-Toeplitz. If original limits remain unchanged under matrix transformations then the infinite matrices are called conservative and the matrices are said to satisfy the conservativeness conditions provided by Kojima – Schur.

Theorem 1.1 (regularity): The matrix $A = (a_{nk})$ is regular if the following conditions are satisfied:

i. $\sum_{k=1}^{\infty} |a_{nk}| < \infty$ for each $n = 1, 2, \ldots$
ii. $\lim\limits_{n\to\infty} a_{nk} = 0$
iii. $\lim\limits_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$

Theorem 1.2 (conservativeness): Let $A = (a_{nk}), n, k = 0,1,2,3, ..$ is an infinite matrix. Then, $A = (a_{nk})$ is conservative matrix if and only if,

1. $\lim_{n\to\infty} a_{nk} = \lambda_k$
2. $\operatorname{Lim}_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = \lambda$
3. $Sup|\sum_{k=1}^{\infty} a_{nk}| \leq M < \infty$ for each $n = 1, 2,$.

Since transforming a sequence and series into another sequence and series which may have more chance to converge by assigning meaningful values to the divergent sequences and series under the given transformation are the main goal of summability theory, this paper focuses on the study of the fixed-point structure of three infinite matrices by classifying each fixed point as either a source or a sink fixed point. Further, stability analysis was performed, designating each fixed point to be either stable or unstable in the underlying sequence space under the given operator. In addition to the study of individual fixed points, we determine the entire fixed set of each transformation: the collection of all points left invariant under the transformation and provide a complete classification of its structure. The fixed set often reveals deeper algebraic and topological properties of the operator and can range from a trivial singleton to an infinite-dimensional subspace. Beyond the fixed-point analysis, we characterise the application of these matrices as linear operators, paying particular attention to their convergent domains. The convergent domain, comprising all sequences whose image under the matrix belongs to a specified target space, is a fundamental object in summability theory. We establish one key topological property concerning their nature as subspaces. This property sheds light on the broader functional-analytic context in which these operators reside and, together with the fixed-point results, contributes to a more complete understanding of the behaviour of infinite matrix transformations.

2. Preliminaries

Definition 2.1: A summability method is a triplet (A,cA(X),limA), where

1. A: DA → F(F,Y) is an application, with DA⊆F(E,X) a vector subspace called the application domain of A.
2. cA(X) is the summability domain defined by cA(X):=v∈ F(E,X): v∈DA and limr→∞A(v)(r) exists.
3. limA:cA(X) → X is the summability operator defined by limA:=limoA|cA(X), where lim:c(E,X) → X is the limit operator at infinity.

Definition 2.2: Let E = F = N. Then a summability method (A,cA(X),limA) is called a matrix-summability method if there exist scalars amn∈C; m,n∈N, such that the application A is given by
A(v)=n≥0am,nvnn≥0v:= (vn)n≥0∈DA

Definition 2.3: Let A:=(A,cA(X),limA) be a kernel-summability method. Let M ⊆ F(E,X) and N ⊆ FF,X be two (vector) subspaces. The method A is called (M,N) conservative if M ⊆ DA and A(M) ⊆ N. We denote by (M,N) the class of (M,N) conservative summability method.

3. Some known results

Corollary 3.1(Nkuno, *et al*, 2025): There exists a modulus operator ≙γ of order γ for every α–circlic porous bounded sequence xnk=1∞ such that ξkk=1∞= ≙γ xii=0∞=γi+1γi ,for each i,

ξn<ξn+1<ξn+2<ξn+3<ξn+4< and

limn→∞ξn→∞.

Lemma 3.1 (Nkuno, *et al*, 2025): Let x=xk be a sequence and let A= ank be an infinite matrix such that
tn= Axn=k=1∞ankxk. Then, the transformation, tn=Axn=k=1∞ankxk defines a mapping f from the sequence x=xk to the sequence tn by fxk=tn=Axn=k=1∞ankxk if and only if A= ank is regular.

Lemma 3.1 (Nkuno, *et al*, 2025): Let x=xk be any summable sequence by the method A= ank. Then, f has a fixed point if and only if A= ank is regular.

Theorem 3.1 (Nkuno, *et al*, 2025): Let xkk=1∞ be an α-porous bounded sequence and ≙γ be the modulus operator defined in corollary 3.1 above. Then, there exist a Riesz matrix

Rp=Rp=1n+1

if k≤n0, otherwise n,k∈N, corresponds to xkk=1∞ if and only if there exist a shift operator, △ such that △(≙γ xk)n=1∞=1nn=1∞

Theorem 3.2 (Nkuno, *et al*, 2025): If A= ank= αnn+1βk -λn ∀ n,k∈N is any matrix transformation, then there exists a sequence of number, γnn=1∞∈0,1 with the property that γn=αn-λnn+1 and limn→∞ γn=1γ such that A= ank is a Toeplitz matrix

Theorem 3.3 (Nkuno, *et al*, 2025): Let A= ank=αkβ-kn+1 for k≥n,n=0,1,2,3,..0, for k≤n be an upper triangular matrix with the property that α,β∈0,1. Then, there exists a sequence of numbers δii=1∞∈0,1 with the property that limn→∞ δn∈0,1 such that A= ank is a Toepleitz matrix.

Theorem 3.4 (Nkuno, *et al*, 2025): The matrix AČ,Br,s=čnkbr,s defined in theorem 4.1 above where

čnkbr,s=ckcn-kcn+1s+rnnksn-krk,0≤k≤n0,

k>n is regular if and only if the series kck cn-k rkk!n-k!sk is summable to Pn where Pn=cn+1s+rnn! sn.

Theorem 3.5 (Nkuno, *et al*, 2025): Let xk be a sequence and let A= ank be a non-identity diagonal Toepleitz infinite matrix. Let f be as defined in the lemma 3.1 above. Then, f has a fixed point if and only if there exist λn>0 for each n such that tn= (Ax)n=k=1∞ank-λnxk
and limn→∞tn=t

Theorem 3.6 (Dotson, 1970) [2]: Suppose E is a real Hilbert space, C is a closed convex subset of E, T: C→C is quasi-nonexpansive on C and has at least one fixed point p∈C, and I-T is demiclosed. Suppose x1∈C and Mx1,A,T is a normal Mann process such that tn = an+1,n+1 is bounded away from 0 and 1. Then the sequences xn, vn converge weakly to a fixed point of T.

Theorem 3.7 (Siddiqui and Nkuno, (2020)): Let A=ank be any infinite matrix and S be the space of all sequences of real or complex numbers. Then, the convergence field FA is dense in SA if and only if the matrix A=ank is irregular.

Theorem 3.8 (Nkuno, *et al*, 2025): Let xkk=1∞ be an α-porous bounded sequence and ≙γ be the modulus operator defined in corollary 3.1 above.
Then, there exists a Riesz matrix

Rp=Rp=1n+1

if k≤n0, otherwise n,k∈N, corresponding to xkk=1∞ if and only if there exist a shift operator, △ such that △(≙γ xk)n=1∞=1nn=1∞

Theorem 3.9 (Nkuno, *et al*, 2025): Let N be the sequence of natural numbers and A= ank be the matrix defined in theorem 3.3. Then, there exist a number δ≥0 such that if f(n)=1 for n=1 n-1 for n>1 is a function on N such that N is α,β-summable to δ then, δ is the fixed point of f if and only if β=1,α=0.

Theorem 3.10 (Nkuno, *et al*, 2025): Let x= xk be an infinite sequence and AČ,Bs,s=čnkbs,s be the infinite matrix defined in theorem 3.4. Then, the first term of the sequence x=xk is fixed under the mapping f: x⟶Axn if and only if the matrix, AČ,Bs,s=čnkss,s is a Toepleitz lower triangular regular matrix.

Theorem 3.9 (Nkuno, *et al*, 2025): Let x= xk be an infinite sequence and AČ,Bs,s=čnkbs,s be the infinite matrix defined in theorem 3.4. Then, the first term of the sequence x=xk is fixed under the mapping f: x⟶Axn if and only if the matrix, AČ,Bs,s=čnkss,s is a Toepleitz lower triangular regular matrix.

4. Main results

Definition 4.1: Let xnn=1∞ be any arbitrary sequence. Then, we shall define a difference operator ∆' on xnn=1∞ as

∆' xn=xn+1-xn.

Theorem 4.1: Let x=xk be any arbitrary sequence and f:N⟶ x=xk be a mapping such that f(n)=xk with a fixed point x*. Further, let A=ank be any of the infinite matrices defined in theorem 3.2-3.4 and let g be a mappings such that Axn=k=1∞ankxk. Then, the fixed point x* is stable (i.e.fgx*=x*) if and only if fx1=x* and the matrix A= ank is a Toeplitz lower triangular infinite matrix.

Proof

Supposed that the fixed point x* is stable. Then, it will follow that fgx*=x* which implies that f-1x*= gx*. Now, if f:N⟶ xk is a mapping such that f(n)=xk with a fixed point x*, then, it will follow that fn*=n*=x*. This implies that x*∈N0 and therefore f-1x*=n*∈N0. Furthermore, f-1x*= gx* which in turn implies that gx*=n*. But then, gx=Axn=k=1∞ankxk which implies that

gx*=k=1∞ankx*=x*.

f-1x*=k=1∞ankx*.

But then, f-1x*=n* which implies that k=1∞ankx*=n*. We know that

fn*=k=1∞ankx*=k=1∞

ankxk where x=xk which also means that k=1∞ankxk=xk=n*. But however, this is possible only if k=1∞ank=1 which is a Toepleitz condition. Further either, n*=x1 or g is an identity mapping. But then, A=ank is a regular matrix. Hence, g is not an identity mapping and therefore n*=x1 which in turn implies that k=1∞ank=x1 . Thus clearly, k=1∞ank=1 and k=1∞ank=x1 implies that A=ank is a lower triangular matrix which consequently implies that A=ank is a Toeplitz lower triangular matrix.
Conversely, supposed that the matrix A= ank is a Toeplitz lower triangular infinite matrix and that f(n1)=n1=x1, then, it will follow that gx1=x1. But then, f(n1)=x1 which implies that gx1=gf(n1)=x1. Now, since x1=n1=x*, and f(x1)=x*, it follow that gf(x*)=x*

Theorem 4.2: Let x=xk be any arbitrary sequence and f:N⟶ x=xk be a mapping such that f(n)=xk with a fixed point x*. Further, let A=ank be any of the infinite matrices defined in theorem 3.2 – 3.4 and let g be a mappings such that g(x)=Axn=k=1∞ankxk. Then, the fixed point x* of f is a sink fixed point if and only if x* is the limit point of x=xk under the mapping g.

Proof

Supposed that x* is the limit point of xk under the mapping g. Then, it will follow that limn→∞tn=x*. But then, tn=Ax*n=k=1∞ankxk. This implies that
limn→∞tn=limn→∞k=1∞ankxk=x*. This implies that k=1∞ankxk=x* which is clearly the sum of the series k=1∞ankxk. This consequently imply that x* is a sink fixed point.
Conversely, supposed that x* is a sink fixed point. Then, it will follows that x*=t where t=limn→∞tn. But then, tn=Ax*n=k=1∞ankxk which consequently, implies that x*=t=limn→∞tn=limn→∞k=1∞ankxk. This implies that fx*=limn→∞k=1∞ankxk. But also, the sequence tn is the transform of the sequence xk. Hence, x* is the limit point of the transformed sequence tn of the sequence xk which consequently implies that x* is a sink fixed point and the put an end to the proof of the theorem.

Theorem 4.3: Let x=xk be any arbitrary sequence of real numbers with fixed point set Fix(A). Let A=ank be any of the operators defined in theorem 3.2 – 3.4. Then, the fixed point x'*∈FixA is sink if and only if limn→∞Axn= x'* otherwise it is source.

Proof

The proof of this theorem follows from theorem 3.9 and theorem 3.10.

Theorem 4.4: Let x=xk be any arbitrary sequence and f:N⟶ x=xk be a mapping such that f(n)=xk with a fixed point x*. Further, let A=ank be any of the constructed infinite matrices defined and let g be a mappings such that Axn=k=1∞ankxk. Then, the fixed point x* is stable (i.e.fgx*=x*) if and only if fx1=x* and the matrix A= ank is a Toeplitz lower triangular infinite matrix.

Proof

Supposed that the fixed point x* is stable. Then, it will follow that fgx*=x* which implies that f-1x*= gx*. Now, if f:N→ xk is a mapping such that f(n)=xk with a fixed point x*, then, it will follow that fn*=n*=x*. This implies that x*∈N0 and therefore f-1x*=n*∈N0. Furthermore, f-1x*= gx* which in turn implies that gx*=n*. But then, gx=Axn=k=1∞ankxk which implies that

gx*=k=1∞ankx*=x*.

f-1x*=k=1∞ankx*.

But then, f-1x*=n* which implies that k=1∞ankx*=n*. We know that fn*=k=1∞ankx*=k=1∞ankxk where x=xk which also means that k=1∞ankxk=xk=n*. But however, this is possible only if k=1∞ank=1 which is a Toepleitz condition. Further either, n*=x1 or g is an identity mapping. But then, A=ank is a regular matrix. Hence, g is not an identity mapping and therefore n*=x1 which in turn implies that k=1∞ank=x1 . Thus clearly, k=1∞ank=1 and k=1∞ank=x1 implies that A=ank is a lower triangular matrix which consequently implies that A=ank is a Toeplitz lower triangular matrix.

Conversely, supposed that the matrix A= ank is a Toeplitz lower triangular infinite matrix and that f(n1)=n1=x1, then, it will follow directly from theorem 3.10 that gx1=x1. But then, f(n1)=x1 which implies that gx1=gf(n1)=x1. Now, since x1=n1=x*, and f(x1)=x*, it follow that gf(x*)=x*

Theorem 4.5: Let S be the space of all real sequences and f:N→S be a mapping with the fixed point x*. Further, let DA and C A be the application and the convergence domain of the infinite matrix A= ank defined in either theorem 3.2 – 3.4 and let g:S→DA be a mapping. Then, the necessary and sufficient condition for x* to be stable is that x* must be the first term of the mapping f:N→S (i.e. x* is the first term of every sequence

x= xk∈S).

Proof

Supposed that x* is the first term of the mapping f:N→S. Then, it will follow from theorem 3.10 that gfx1=gx1=k=1∞an1x1. But then, the matrix A= ank is a Toeplitz lower triangular infinite matrix. Hence,

gx1=k=1∞an1x1 =x1 = x*

k=1∞an1=1 (a Toeplitz condition for regularity)

gx1=x*=fx1

This implies that the fixed point x* remained the same throughout and hence stable. This now, prove the result of the theorem.

Theorem 4.6: Let x=xk be any arbitrary sequence, and f:N→ x=xk be a mapping such that f(n)=xk with a fixed point x*. Further, let A=ank be any of the infinite matrices defined in theorem 3.2-3.4 and let g be a mappings such that g(x)=Axn=k=1∞ankxk. Then, the fixed point x* of f is a sink fixed point if and only if x* is the limit point of x=xk under the mapping g.

Proof

Supposed that x* is the limit point of xk under the mapping g. Then, it will follow that limn→∞tn=x*. But then, tn=Ax*n=k=1∞ankxk. This implies that

limn→∞tn=limn→∞k=1∞ankxk=x*

This implies that k=1∞ankxk=x* which is clearly the sum of the series k=1∞ankxk. This consequently imply that x* is a sink fixed point.

Conversely, supposed that x* is a sink fixed point. Then, it will follow that x*=t where t=limn→∞tn. But then, tn=Ax*n=k=1∞ankxk which consequently implies that x*=t=limn→∞tn=limn→∞k=1∞ankxk. This imply that fx*=limn→∞k=1∞ankxk. But also, the sequence tn is the transform of the sequence xk. Hence, x* is the limit point of the transformed sequence tn of the sequence xk which consequently implies that x* is a sink fixed point and the put an end to the proof of the theorem.

Theorem 4.7: Let S be the space of all sequences and f:N→ xk be a mapping with the fixed point x*. Further, let DA and CA be the application and the convergence domain of the infinite matrix A= ank defined above and let g and h be two mappings under the transformation Axn=k=1∞ankxk such that g:S→DA and h:DA→CA. Then, the fixed point x* is:

Stable if and only if f(gx*)=x*

Completely stable if and only if fnr=hxr=gtr=x*

Proof

If we consider the fixed point of the mapping f:N→ xk to be x1, then, it will imply that fn1=x1=x* which will in turn follow that gfx*=x* and the matrix A= ank is a Toeplitz lower triangular infinite matrix and hence the proof of the theorem. But if on the other hand, we consider that x1 is not the fixed point of the mapping f:N→ xk i.e. fn1≠x1=x* but rather nr such that fnr=nr=x*, then, it will follow that gfnr=gnr=k=1∞ankxk=tr. Now, supposed that nr is the median term xM of the sequence x=xk. Then, it will follow that gnr=gxM=xM=x* which implies that gfx*=x* and hence x* is stable. Further, if we take nr not to be the median term xM of the sequence x=xk, then, it will follow that gfnr=gnr=k=1∞ankxk=tr. But then, what is tr ? it is x* or something else? If tr=x*, then, the result follows but if something else (i.e.tr≠x*), then, gfx*≠x*and hence, x* is unstable.

Conversely, supposed that gfx*=x*, the, it will follow that gx*=x* which implies that fnr=x*. But then, fnr=xr. Hence, xr=x*

k=1∞ank fnr=k=1∞ankxr=k=1∞ankx*.

But however,

k=1∞ankxk=tn

k=1∞ankx*=tr

x*k=1∞ank=tr

But then,

gx=k=1∞ankxk=x*,

x*k=1∞ank=tr= x*

x*k=1∞ank=x*

k=1∞ank=1

This implies that the matrix A= ank is regular which in turn implies that the graph GA of gx and the graph GN of fx intersect the graph y=n at the point x*. Hence, the fixed point x* is stable.

ii. Supposed that the fixed point x* is completely stable, then, it will follow that hgfx*=x*. This implies that gfx*=x* which in turn implies that fx*=x*. This will consequently mean that the graph GhA of ht, GgA of gx and GfN of fn all intersect the graph y=n at the point x*. But then, gxr=x*, htr=x* and fnr=x* which in turn implies that

htr=gxr=fnr=x* (i.e.hgfnr=x*).

Conversely, supposed thatgfnr=x*). Then, it will imply that hgxr=x* and fnr=x* which will in turn implies that fnr=x*,hxr=x*,and gtr=x*. Thus clearly, htr=gxr=fnr=x*which consequently, mean that the graph GhA of ht, GgA of gx and GfN of fn all intersect the graph y=n at the point x*. This shows clear consistency among all the graphs at the point x*

which implies that the functions f,g,and h are all stable at x* and hence, the fixed point x* of the mapping f:N→ xk is completely stable under the transformation Axn=k=1∞ankxk governed by the mappings g and h

Theorem 4.8: Let S be the space of all real sequences and let A= ank be any of the constructed matrices such that Axn=k=1∞ankxk. If suppose that DA and CA are the application and the convergence domain of A= ank. Then, the transformation Axn=k=1∞ankxk defines a mapping f:S→DA and g:DA→CA such that gfx=tn if and only if g is an identity mapping.

Proof

Suppose that for each x=xk∈S, there is a mapping f:S→DA defined by fxk=tn where tn=(Ax)n=k=1∞ankxk. Further, let g:DA→CA be a mapping defined by gtn=zi where zi=(At)i=j=1∞aijtj. But then, CA⊂DA⊂S which implies that each zi∈CA is also contained in DAthat is to say zi∈DA also. More so, A= ank is a Toepleitz matrix implies that CA⊆DA and therefore, every sequence zi∈CA also belongs to DA. This implies that the matrix A'=aij=1 which shows that zi=tn. Thus we have

gfx=gAxn=gtn=tn=fx for every fx∈DA

Conversely, suppose that g is an identity mapping, then, it will follow that that the mapping g:DA→CA is an isomorphism which means that g is one to one and onto mapping. This in turn implies that CA is very where dense in DA which simply implies that each sequence zi∈DA is also in CA i.e. zi∈CA. But then, Axn=tn=k=1∞ankxk is a transformation from S to DA. Let f be such a transformation defined by fx=Axn=tn=k=1∞ankxk. Now, since g is an identity mapping from DA to CA, then, it will follow that

gfx=gAxn=gtn=tn

which implies the transformation Axn=k=1∞ankxk defines a mapping f:S→DA and g:DA→CA such that gfx=tn if and only if g is an identity mapping and this proves the result

Theorem 4.9: Let S be the space of all real sequences and let A= ank be any of the constructed infinite matrices such that Axn=k=1∞ankxk. Let DA and CA be the application and the convergence domain of A= ank. Then, the transformation Axn=k=1∞ankxk defines a mapping
f:S→DA and g:DA→CA such that f=g on DA if and only if A= ank is regular.

Proof

Suppose that f=g on DA, then, it will follow that fx= Axn=tn=k=1∞ankxk=gx for some x∈DA. Now, since CA is everywhere dense in DA by theorem 3.7, it follows that g is a bijective mapping. Further, since g:DA→CA and f:S→DA, it follows that f:S→CA. Hence, each sequence fix= xi'∈DA is also in CA i.e. xi'∈CA. But then, this is possible only if A= ank is regular.

Conversely, suppose that A= ank is regular. Then, it will follow from theorem 3.7 that CA is dense in DA. Now, if g is any mapping from DA to CA, then, it will follow that g is a bi-jective mapping which means that g is one to one and onto. This implies that for every y∈DA there exist z∈C such that z=gy=y. Moreover, every y∈DA is of the form y=k=1∞ankxk and f is a mapping from S to DA of the form

fx=Axn=k=1∞ankxk

for some x∈S which implies that y=fx=Axn=k=1∞ankxk=y∈DA and since C is an everywhere dense subset of DA, it follows that every y∈DA is also in CA. But then, we know that the elements of CA are of the form gy = y=k=1∞ankxk which means that fx=gy and thus f=g if and only if x=y as required

Theorem 4.10: Let S be the space of all real sequences x= xk and Ω be the space of all infinite matrices A=ank such that Axn=k=1∞ankxk. Let f:S→SΩ be a transformation defined by fx= Axn=k=1∞ankxk for all A∈Ω,x∈S where A= ank and x= xk. Then, f has a fixed point if and only if every A∈Ω is regular. Further, C0, C1, C are fixed subsets of CΩ and C0 ⊂C1⊂C⊂CΩ if and only if A=ank is regular.

Proof

We know that if A=ank∈Ω is any regular matrix and x= xk∈S is any arbitrary divergent sequence such that fx= Axn=k=1∞ankxk, then, it will follow by Lemma 3.2 that f has fixed point. Further, since the matrix A=ank and x= xk is arbitrary, it will follow that for each A=ank∈Ω and each x= xk∈S, f has fixed point. Now, denote by x0,x1,x2, ..,xk, .. the set of those fixed points. Thus if S is the space of all sequences xk and Ω is the space of all infinite regular conservative matrices A=ank such that Axn=k=1∞ankxk, then, it will follow that the mapping f:S→SΩ such that
fx= Axn=k=1∞ankxk for all A∈Ω,x∈S has the fixed point x0,x1,x2, ..,xk, .. which is common to them.

Conversely, suppose that that the mapping f:S→SΩ given by fx= k=1∞ankxk for all A∈Ω,x∈S has the common fixed points x0,x1,x2, ..,xk, .., then, it will follow that the graphs of fx=k=1∞ankxk intersect the graph of y=x which implies that for each A∈Ω,x∈S the graph of fx intersect the graph of y=x.

This is possible only if $\lim_{n\to\infty} fx = \lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}x_k = t$. This signifies that $A=a_{nk}$ is regular but $\lim_{k\to\infty} x_k = \lim_{n\to\infty} fx = \lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}x_k = t$ for some, $x=x_k \in S$ which implies that $A=a_{nk}$ is regular

Theorem 4.11: Let S be the space of all real sequences and $A= a_{nk}$ be an infinite matrix such that $fx=Ax_n=\sum_{k=1}^{\infty} a_{nk}x_k$. Let C_A be the convergence domain of $A= a_{nk}$ and C the space of all convergence sequence. Then, the set C_A is an extension field of the space C if and only if $A= a_{nk}$.

Proof
We know that for all $x\in C$ where $x=x_k$, we have $\lim_{k\to\infty} x = \lim_{k\to\infty} x_k = x'$. Further, we also know that for every arbitrary $x\in C_A$, we have $\lim_{n\to\infty} t_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}x_k = t$ where $x=x_k$. But then, this is possible only if $A= a_{nk}$ is regular.
Conversely, suppose that $x\in C_A$ and $A= a_{nk}$ is regular. Then, it will follow that e $\lim_{n\to\infty} t_n = t$ where $t_n=Ax_n= \sum_{k=1}^{\infty} a_{nk}x_k$. But then, $\lim_{k\to\infty} x_k$ may or may not exist. If it exist, then, $\lim_{k\to\infty} x_k= \lim_{n\to\infty} t_n$ and $x\in C_A$. Hence, C_A is dense in C_A. But if $\lim_{k\to\infty} x_k$ does not (since $x=x_k$ is arbitrary), then, it will follow that $\lim_{k\to\infty} x_k \neq \lim_{n\to\infty} t_n$ which implies that the sequence $x=x_k$ does not belong to C_A. But however, for all $x\in C$, $x\in C_A$ since $\lim_{k\to\infty} x_k= \lim_{n\to\infty} t_n$. Hence, $C\subset C_A$ and this ends the proof of the theorem.

Theorem 4.12: Let S be the space of all real or complex sequences and let $A= a_{nk}$ be one of the constructed infinite matrices such that $Ax_n=\sum_{k=1}^{\infty} a_{nk}x_k$. Further, let D_A and C_A be the application and the convergence domain of $A= a_{nk}$ such that $f:D_A\longrightarrow C_A$ is a mapping defined by $fx= Ax_n=\sum_{k=1}^{\infty} a_{nk}x_k$. Then, the set C_A is fixed in D_A if and only if the matrix $A=(a_{nk})$ is regular.

Proof
To prove this result, let us suppose first that the set C_A is fixed in D_A. Then, it will follow that for every $x_n\in C_A\subset D_A$

$$fx_n=x_n$$

But then,

$$x_n=Ax_n=\sum_{k=1}^{\infty} a_{nk}x_k$$

$$fx_n=x_n=Ax_n=\sum_{k=1}^{\infty} a_{nk}x_k$$

$$fx_n=\sum_{k=1}^{\infty} a_{nk}x_k$$

But then, we know that every $t_n\in D_A$ is of the form

$$t_n=\sum_{k=1}^{\infty} a_{nk}x_k \text{ for each } n$$

$$t_n=\sum_{k=1}^{\infty} a_{nk}x_k$$

$$fx_n=t_n=\sum_{k=1}^{\infty} a_{nk}x_k\in D_A$$

But $fx_n=x_n$ and $fx_n=t_n$

$$f^{-1}t_n=x_n$$

$$f^{-1}t_n=fx_n=x_n$$

$$t_n=f^2x_n=fx_n=x_n$$

$$t_n=x_n$$

$$f=1$$

This implies that the matrix $A=a_{nk}$ is conservatively regular under f.
Conversely, suppose that the matrix $A=a_{nk}$ is regular. Then, it will follow that

$$\lim_{n\to\infty} t_n=x$$

if and only if

$$\lim_{k\to\infty} x_k=x$$

$$\lim_{k\to\infty} x_k=\lim_{n\to\infty} t_n=\lim_{n\to\infty} Ax_n=\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}x_k$$

$$\lim_{k\to\infty} x_k=\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}x_k$$

$$\lim_{k\to\infty} x_k=\lim_{n\to\infty} Ax_n=\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}x_k$$

$$\lim_{k\to\infty} x_k=\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}x_k$$

$$x_k=\sum_{k=1}^{\infty} a_{nk}x_k=t_n$$

This implies that the mapping given and defined by

$$fx= Ax_n=\sum_{k=1}^{\infty} a_{nk}x_k$$

for every $x\in D_A$

$$fx=t_n$$

$$fx=x$$

This clearly shows that x_k is fix. But then, x_k is arbitrary. Hence, the space C_A is fixed in D_A and this end the proof of the theorem

Theorem 4.13: Let S be the space of all real or complex sequences and $A= a_{nk}$ be an infinite matrix such that $Ax_n=\sum_{k=1}^{\infty} a_{nk}x_k$. Let C_A be the convergence domain of $A= a_{nk}$. Then, the set C of all convergence sequence is fixed in C_A if and only if the matrix $A=(a_{nk})$ is regular.

Proof
Supposed that the set C of all convergence sequence is fixed in C_A, then, it will follow that for all $x\in C_A$ where $x=x_k$ we have $fx=x_k$. But then, $fx=Ax_n=\sum_{k=1}^{\infty} a_{nk}x_k$

$$fx=x=Ax_n=\sum_{k=1}^{\infty} a_{nk}x_k$$

$$x_k=\sum_{k=1}^{\infty} a_{nk}x_k$$

$$x_k=\sum_{k=1}^{\infty} a_{nk}x_k$$

But then,

$$\lim x=\lim_{k\to\infty} x_k=x' \ \forall\ x=x_k\in C \text{ and}$$

limn→∞Axn=limn→∞k=1∞ankxk=limn→∞tn=t ∀ x∈CA

Furthermore, C⊂CA and that for all x∈CA, we have

fx=Axn=k=1∞ankxk

This implies that every x∈C is of the form

fx=Axn=k=1∞ankxk

Therefore, for all x∈CA, we have

limx=limk→∞xk=limn→∞

fx=limn→∞Axn=limn→∞k=1∞ankxk=limn→∞tn=t

But then,

limx=limk→∞xk=x' ∀ x=xk∈C

limx=limk→∞xk=x'=limn→∞tn=t

limk→∞xk=limn→∞tn

x'=t

This implies that the matrix A= ank is conservatively regular. Conversely, supposed that the matrix A= ank is conservatively regular. Then, it will follow that for all x∈CA where x=xk, we have limk→∞xk=x' to imply that limn→∞tn=x' where tn=Axn=k=1∞ankxk. This implies that

limk→∞xk=limn→∞tn=limn→∞k=1∞ankxk.

But then, we know that every point x∈C is of the form

fx=Axn=k=1∞ankxk.

This implies that

limk→∞xk=limn→∞tn=limn→∞k=1∞ankxk=limn→∞Axn=limn→∞ fx=x'

limk→∞xk=limn→∞ fx

But then, recall that x=xk.
Therefore,

limk→∞x=limk→∞xk=limn→∞ fx

x= fx

fx=x

Thus the space C of all convergence sequence is fixed in CA the space of convergence domain of the matrix A= ank which now proves the theorem

Theorem 4.14: Let S be the space of all real or complex sequences x= xk and A=ank be an infinite matrices. Let g:DA→CA be a mapping defined by gx= Axn=k=1∞ankxk for some x∈S where A= ank and x= xk. Then, g is isometric if and only if the matrix A=ank is regular.

Proof
Suppose that the mapping g :DA→CA is an isometric mapping. Then, it follows that for all x,xo∈DA there exist gx,gxo∈CA such that
ϱψx,ψxo=ρx,xo where ϱ and ρ are metrics on CA and DA respectively.
Now, take xo∈DA such that gxo∈FA. Then, given εCA>0 ∃,δDA>0 such that
ϱgx,gxo≤ εCA to imply that ρx,xo≤δDA
⇔Given εCA>0 ∃,δDA>0 such that
gx∈Bgxo, εCA whenever x∈Bxo,δDA.
This implies that if G is any open set in SA, then, given εFA>0 ∃,δSA>0 such that

g Bxo,δDA⊂Bgx, εCA⊂G

gBxo,δDA⊂G

g-1og Bxo,δDA⊂g-1G

Bxo,δDA⊂g-1(G)

g-1og=1 on DA

CA ⊆ DA

This implies that g-1 is open in DA and since g-1og=1, ψ is a one to one mapping.
Further, let g:DA→CA be any mapping and let g(G) ⊂CA be the corresponding image of any open set G⊂ DA. Then, if we suppose that gG=∅, then, it will obviously follow that gG is open in CA. Therefore, let us suppose on the other hand that GG≠∅ and let xo∈gG be arbitrary such that g-1(xo)∈G. Then, since G is open in DA, there exist εDA>o such that Bg-1xo, εDA⊂G. Further, given εDA>o,∃ δDA>o such that

g-1 Bxo,δDA⊂Bg-1xo, εCA⊂G

g-1 Bxo,δDA⊂G

go g-1 Bxo,δDA⊂gG

Bxo,δDA⊂g(G)

g-1og=1 on CA

This implies that ψ(G) is open in FA and since g-1og=1, it follows that g is an onto mapping.
This implies that CA ⊇ DA
Thus by (1.3) and (1.6), we have
CA= DA
But then, Pxo,CA=limδ→0λxo,δ,CAδ
limδ→0λxo,δ,CAδ=0
⟹λxo,δ,CA=ϕ
But also
(xo,δ,CA)= Supα>0:∃x∈Bxo,δ Bx,α⊂Bxo,δ and Bx,α⋂CA=∅
⟹ Supα>0:∃x∈Bxo,δ Bx,α⊂Bxo,δ and Bx,α⋂CA=∅=ϕ
This implies that there does not exist any number α>0 for any given arbitrary point xo∈ DA such that Bxo,α⋂CA=∅. Therefore, since such a number α>0 does not exist for any

arbitrary point $x_o \in D_A$ such that $B(x_o,\alpha) \cap C_A = \emptyset$, it follows that the matrix $A=(a_{nk})$ is regular.
Conversely, suppose that the matrix $A=(a_{nk})$ is regular. Then, it will follow that all the sequence $(t_n)_{n=1}^{\infty}$ for which the series $\sum_{k=1}^{\infty} a_{nk}x_k$ exist and converge, converges. This simply means that C_A is dense D_A. Therefore, for any arbitrary point $x_o \in D_A$ there does not exist any number $\alpha>0$ such that

$$B(x_o,\alpha) \subset D_A \text{ and } B(x_o,\alpha) \cap C_A = \emptyset.$$

Now, let $\psi: S_A \to F_A$ be any continuous mapping and let G be any open set in D_A. Then, since for any $x_o \in D_A$ arbitrary, there does not exist any number $\alpha>0$ such that $B(x_o,\alpha) \subset D_A$ and $B(x_o,\alpha) \cap C_A = \emptyset$, it follows that given $\varepsilon_{C_A}>0$ there exist $\delta_{D_A}>0$ such that

$$g^{-1}\left(B(x_o,\delta_{D_A})\right) \subset B(g^{-1}(x_o), \varepsilon_{C_A}) \subset G$$

$$g^{-1}\left(B(x_o,\delta_{D_A})\right) \subset G$$

$$g o g^{-1}\left(B(x_o,\delta_{D_A})\right) \subset g(G)$$

$$B(x_o,\delta_{D_A}) \subset g(G)$$

$$g o g^{-1} = 1 \text{ on } C_A$$

This implies that g is an onto mapping and hence $C_A \supseteq D_A$. Moreover, since the matrix $A=(a_{nk})$ is irregular, it follows that $C_A \subseteq D_A$. This implies that given any $g(x_o) \in C_A, \exists x_o \in D_A$ such that given $\varepsilon_{C_A}>0$ there exist $\delta_{dA}>0$ such $B(x_o,\delta_{D_A}) \subset D_A$ and that $B(g(x_o), \varepsilon_{C_A}) \subset C_A$.
This implies that there exist $\delta_{D_A}>0$ for any given $\varepsilon_{C_A}>0$ such that
$\rho(x,x_o)<\delta_{D_A}$ to imply that $\varrho(g(x),g(x_o))< \varepsilon_{C_A}$.

$$\varrho(g(x),g(x_o)) \leq \rho(x,x_o).$$

But then, the matrix $A=(a_{nk})$ is regular. Hence, $\varrho(g(x),g(x_o)) = \rho(x,x_o)$. This implies that g is Isometric.

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