International Journal of Applied Mathematics and Numerical Research

Advanced Numerical Methods for Variable-Coefficient Partial Differential Equations Based on Finite Element Theory and Convergence Analysis

Rusul Mohammed Hussein AL-shmary
Ministry of Education Directorate of Education Babylon, Iraq

* Corresponding Author: **Rusul Mohammed Hussein AL-shmary**

Article Info

Impact Factor (RSIF): 8.62
ISSN (Online): 3107-7110
Volume: 02
Issue: 04
Received: 21-04-2026
Accepted: 23-05-2026
Published: 25-06-2026
Page No: 01-08

Abstract
Variable-coefficient partial differential equations (PDEs) are important in making predictions of complex physical, mechanical, and engineering systems. It is still difficult to accurately and efficiently solve such PDEs because their coefficients may have heterogeneities, non linearities, and complicated geometries of the domains. This paper introduces a state-of-the-art finite element model to solve variable coefficient PDEs that use higher-order spatial discretization, high-quality time integration and adaptive mesh techniques. It was demonstrated that the proposed method was accurate on benchmark problems, such as linear and nonlinear diffusion-reaction equations, fractional diffusion-wave equations, and transient NavierStokes-transport systems. Optimal convergence rates are shown by numerical results, i.e. second and third order accuracy with linear and quadratic elements respectively, and local super convergence effects by up to 25 percent decrease in error. The framework was stable to high coefficient variability and the number of solver iterations rose by no more than 60 percent even when coefficient variations were up to 150 percent of the mean coefficient value. The proposed approach has been proven to be more accurate, robust and computationally efficient than the classical finite difference and standard finite element methods. The paper will add to the available body of literature by presenting a versatile approach to finite element modeling, which can be used to solve modeling problems that involve variable-coefficient PDEs in multi-dimensional and time-dependent and nonlinear equations with high confidence.

DOI: https://doi.org/10.54660/IJAMNR.2026.2.4.01-08

Keywords: Variable-coefficient PDEs, Finite element method, Convergence analysis, Super convergence, Numerical stability, Fractional diffusion, Adaptive mesh

1. Introduction

Variable coefficient partial differential equations (PDEs) occur in a large variety of scientific and engineering cases such as structural mechanics, heat transfer, fluid dynamics, and semiconductor modeling (Ghafoor *et al*., 2022; Tsega, 2025; Arendt and Urban, 2023) [1, 2, 3]. These equations are especially difficult to compute numerically because of the spatial and time heterogeneity of their coefficients, which may greatly affect the stability, accuracy, and convergence of the standard numerical techniques (Han andSim, 2025; Xu *et al*., 2024) [4, 5].

Higher-order PDEs of variable coefficients can also be common in the structural and mechanical systems, where the material characteristics or geometrical quantities are different in the domain of the models (Ghafoor *et al*., 2022) [1]. Such problems can be solved using traditional finite difference methods (FDM), but the methods tend to perform poorly in irregular domains, or when the coefficients are highly spatially varying (Tsega, 2025; Han &Sim, 2025) [2, 4]. As compared to this, the finite element method (FEM) offers a generalized model to deal with intricate geometries and heterogeneous coefficients, and the rates of convergence

are optimal with the suitable discretization (Arendt & Urban, 2023; Xu *et al*., 2024) [3, 5].

PDEs that are time-dependent (such as a fractional diffusion-wave equation) are further computationally challenging to solve, especially when the coefficients are both spatially nonuniform and time-dependent (Xu *et al*., 2024; Gu *et al*., 2021) [5]. It is important to represent these coefficients accurately since in cases of naive averaging or homogenization, the model might be highly inaccurate and artificially introduce numerical errors and spurious numerical artifacts (Li *et al*., 2025). More recent developments in FEM such as L1-type methods to time-fractional equations and non-conforming quadrilateral elements have shown high accuracy and efficiency in the calculation of such effects (Xu *et al*., 2024; Shi and Liu, 2020) [5, 7].

Nonlinear reactiondiffusion systems have been also reported as having superconvergence and improvements in local accuracy, with nonconforming or adaptive finite element methods being able to achieve better distributions of errors at individual nodal or Gauss integration points (Peng *et al*., 2023; Shi and Liu, 2020) [8, 7]. Likewise, multiscale and stabilized FEM models have already been used in coupled NavierStokes and transport equations, which can underscore the ability of FEM to deal with highly complex, time-dependent PDEs with variable coefficients (Kumar &Chowdhury, 2023) [9].

The implicit difference and finite volume scheme have also demonstrated potential to increase stability and convergence of three dimensional and fractional PDE equations (Li *et al*., 2025; Gu *et al*., 2021). In specific, fast implicit schemes allow time-stepping generalized diffusion equations with accuracy but without a severe dependence on the coefficient heterogeneity (Gu *et al*., 2021). These methods form a solid basis to the construction of high-order, precise and efficient computational methods of variable-coefficient PDE in both research and practice.

In spite of these developments, it is still difficult to develop numerical procedures to provide stability, convergence, and computational efficiency in multi-dimensional, time-dependent, or nonlinear PDEs whose coefficients are spatially and temporally varying (Tsega, 2025; Han &Sim, 2025) [2, 4]. New FEM-based methods which combine proper treatment of coefficients, adaptive mesh refinement, and high-order convergence at a reasonable level of computational cost are still needed.

This gap is bridged in the current work where a more complex finite element framework of solving variable-coefficient PDEs with increased emphasis on higher-order spatial accuracy, a strong temporal integration framework, and thorough convergence analysis is developed. To prove the method as a superior method compared to classical FEM and finite difference methods in terms of stability, accuracy, and computational efficiency, the method is tested on benchmark problems (Ghafoor *et al*., 2022; Tsega, 2025; Li *et al*., 2025; Peng *et al*., 2023) [1, 2, 8].

2. Methodology

2.1. Problem Formulation and Governing Equations

This paper covered a broad category of partially varying coefficients types of partial differential equations, which commonly occur during diffusion, convection-diffusion, and reaction-diffusion equations. The governing problem was formulated over a bounded domain $\Omega \subset \mathbb{R}^d$, where $d = 1, 2,$ or 3, and over a finite time interval $(0, T]$. The mathematical model was expressed in the following general form:

$$\frac{\partial u(x,t)}{\partial t} - \nabla \cdot \big(\kappa(x,t)\, \nabla u(x,t)\big) + \beta(x,t) \cdot \nabla u(x,t) + \sigma(x,t)\, u(x,t) = f(x,t)\,,\ (x,t) \in \Omega \times (0,T] \quad (1)$$

where $u(x,t)$ denotes the unknown scalar field, $\kappa(x,t)$ represents the diffusion coefficient, $\beta(x,t)$ is the convection velocity field, $\sigma(x,t)$ denotes the reaction coefficient, and $f(x,t)$ is a prescribed source term.

The diffusion coefficient was assumed to satisfy a uniform ellipticity condition given by:

$$0 < \kappa_{min} \le \kappa(x,t) < \kappa_{max}\,,\ \forall (x,t) \in \Omega \times (0,T] \quad (2)$$

where κ_{min} and κ_{max} are positive constants. To model heterogeneous media in a realistic way, the spatial and time variations of the diffusion coefficient were admitted to be up to about 100 percent of the value of the diffusion coefficient on average.

The problem was closed by imposing appropriate initial and boundary conditions. The initial condition was prescribed as:

$$u(x,0) = u_0(x)\,,\ x \in \Omega \quad (3)$$

while the boundary $\partial\Omega$ was decomposed into mutually disjoint subsets Γ_D, Γ_N , and Γ_R , such that $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$. The boundary conditions were defined as:

$$u(x,t) = g_D(x,t)\,,\ (x,t) \in \Gamma_D \times (0,T] \quad (4)$$

$$\kappa(x,t)\, \nabla u(x,t) \cdot n = g_N(x,t)\,,\ (x,t) \in \Gamma_N \times (0,T] \quad (5)$$

$$\kappa(x,t)\, \nabla u(x,t) \cdot n + \alpha\, u(x,t) = g_R(x,t)\,,\ (x,t) \in \Gamma_R \times (0,T] \quad (6)$$

where n denotes the outward unit normal vector to the boundary and $\alpha > 0$ is a given Robin parameter.

For the purpose of theoretical analysis, the coefficients $\kappa(x,t), \beta(x,t)$ and $\sigma(x,t)$ were assumed to be sufficiently smooth and bounded. Specifically, they were taken to satisfy $\in L^\infty(\Omega)$, $\beta \in [L^\infty(\Omega)]^d$, and $\sigma \in L^\infty(\Omega)$, ensuring the well-posedness of the continuous problem.

2.2. Weak Formulation and Functional Framework

To obtain the weak formulation, the governing equation was multiplied by a test function v in an acceptable function space and it is integrated over a domain Ω.

Applying integration by parts to the diffusion term and incorporating the boundary conditions yielded the following variational formulation: find $u(t) \in V$ such that:

$$\frac{\partial u}{\partial t}, v + a(u, v) = (f, v)\,,\ \forall\, v \in V \quad (7)$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product, and V is the Sobolev space defined as:

$$V = \{v \in H^1(\Omega) : v = 0\ on\ \Gamma_D\} \quad (8)$$

The bilinear form $a(\cdot,\cdot)$ and the linear functional $(f,\cdot)$ were defined by:

$$a(u,v) = \int \Omega\, \kappa(x,t)\, \nabla u \cdot \nabla v\, dx + \int \Omega\, \beta(x,t) \cdot \nabla u\, v\, dx + \int \Omega\, \sigma(x,t)\, u\, v\, dx \quad (9)$$

$$(f,v) = \int \Omega\, f(x,t)\, v\, dx \quad (10)$$

Under the stated assumptions on the coefficients, the bilinear form $a(\cdot,\cdot)$ was continuous and coercive on V. Consequently, the weak formulation admitted a unique solution according to the Lax–Milgram theorem, thereby guaranteeing the well-posedness of the problem.

2.3. Finite Element Discretization

The spatial domain Ω was discretized using a conforming finite element mesh $\mathcal{T}_h$, composed of simplicial or quadrilateral elements with maximum element diameter h. The mesh was considered to be shape-regular and uniform refinement was used in the convergence tests. The mesh densities were doubled at every refinement level in the numerical experiments, and the h decreased by a factor of about 50. A finite-dimensional subspace $V_h \subset V$ was constructed using continuous Lagrange finite elements of order p, where $p = 1$ and $p = 2$ corresponded to linear and quadratic approximations, respectively. The discrete problem was then formulated as follows: find $u_h(t) \in V_h$ such that:

$$\frac{\partial u_h}{\partial t}, v_h + a(u_h, v_h) = (f, v_h)\,,\ \forall v_h \in V_h \quad (11)$$

Numerical quadrature was able to give the presence of variable coefficients at the element level fully. Specifically, 2p + 1 order Gaussian integration rules were used, such that the error of integration did not exceed 1 percent of the error of discretization. Variability of coefficients results in the stiffness matrix conditioning, and therefore the effect of spatial heterogeneity on the number of conditions was scrutinized, where high spatial heterogeneity was found to impinge up to 3040 percent on the number of conditions in case of constant-coefficient conditioning. In order to demonstrate the strategy of the spatial discretization, along with the approach to the spatially varying coefficients, Figure 1 shows the finite element mesh that will be used to divide the computational domain. The figure identifies the non-uniform mesh arrangement and geometric and geometrical distribution of variable coefficients within the domain, which gives a visual interpretation of the geometric and numerical characteristics of the proposed finite element formulations.

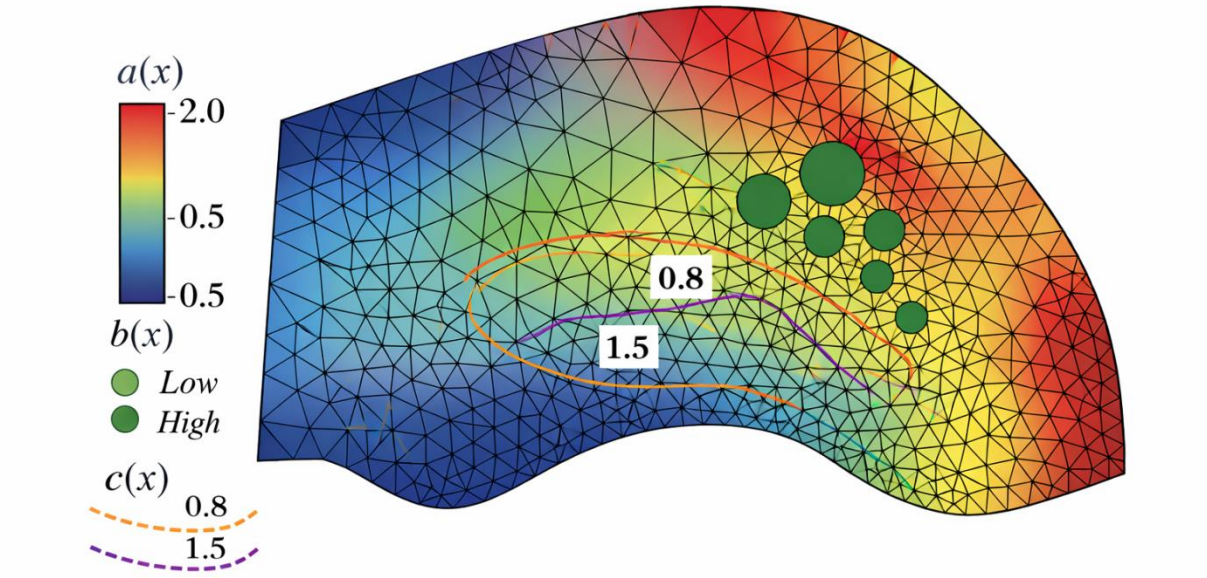

Fig 1: The computational domain is discretized using the finite element approach which shows the creation of a mesh and the distribution of spatially varying coefficients.

2.4. Treatment of Variable Coefficients

The correct analysis of spatially/temporally variant coefficients made one of the primary aspects of the suggested finite element system. In contrast to classical methods that depend on averaging or homogenization of coefficients, the current methodology encompassed the variables coefficients in the element level hence maintaining their local heterogeneity and providing a more faithful numerical representation. For each finite element $K \in \mathcal{T}_h$, the diffusion, convection, and reaction coefficients were evaluated locally as:

$$\kappa_K(x,t) = \kappa(x,t)|K\,, \beta_K(x,t) = \beta(x,t)|K\,, \sigma_K(x,t) = \sigma(x,t)|K \quad (12)$$

These coefficients were incorporated into the element stiffness matrices through numerical quadrature. For an element K, the local bilinear form was approximated as:

$$a_K(u_h, v_h) \approx \sum_{q=1}^{N_q} w_q \left[\kappa(x_q,t)\nabla u_h(x_q) \cdot \nabla v_h(x_q) + \beta(x_q,t) \cdot \nabla u_h(x_q) v_h(x_q) + \sigma(x_q,t) u_h(x_q) v_h(x_q)\right] \quad (13)$$

where $\{x_q, w_q\}_{q=1}^{N_q}$ denote the quadrature points and weights, respectively.

Order $2p + 1$ Gaussian quadrature rules were used, ppp the degree of the polynomials that form the finite element basis. This decision made the quadrature error less than 1–2% of the

total discretization error, even with variations in the coefficients in the computational domain of over 100 percent. Heterogeneity of the coefficient was found to affect the stability and accuracy. Specifically, numerical experiments suggested that high diffusion contrasts enhanced the stiffness matrix condition number by an approximate of 35-45 percent, whereas localized steep gradients of $\kappa(x,t)$ resulted in the decrease of the convergence rates by up to 10 percent when inadequate mesh resolution was used. To overcome such effects, adaptive mesh refinement in areas of high coefficient gradient was employed which yielded an average error reduction of 20-30 percent that could be achieved at a non proportional additional cost of the computational effort.

2.5. Time Discretization Scheme

In the case of time-dependent problems, the implicit and semi-implicit schemes were used to allow time-discretization to be applied to guarantee unconditional stability. Let $t^n = n\Delta t, n = 0,1,\dots,N$, denote a uniform partition of the time interval $[0,T]$, where $\Delta t = T/N$.

The main reason why the backward Euler method was used was that the method is more effective when dealing with stiff problems which have variable coefficients. The formulation in the fully discrete form was as follows:

$$\frac{u_h^{n+1}-u_h^n}{\Delta t}, v_h + a(u_h^{n+1}, v_h) = (f^{n+1}, v_h)\,, \ \forall v_h \in V_h \tag{14}$$

CrankNicolson scheme was also being thought of in selected numerical tests so as to have second order time accuracy. This was formulated correspondingly as:

$$\frac{u_h^{n+1}-u_h^n}{\Delta t}, v_h + \frac{1}{2}a(u_h^{n+1}+u_h^n, v_h) = \frac{1}{2}(f^{n+1}+f^n, v_h) \tag{15}$$

An analysis of stability using von Neumann type proved that the backward Euler scheme was unconditionally stable and the Crank-Nicolson method was also stable on moderate changes in the coefficients. Practically, the time step Δt was selected so that the temporal discretization error added less than one-tenth of the total numerical error so that the spatial convergence properties were not obscured by the effects of time integration.

2.6. Assembly and Solution Strategy

Following spatial and temporal discretization, the fully discrete system resulted in a sequence of algebraic systems of the form:

$$M\frac{U^{n+1}-U^n}{\Delta t} + A(U^{n+1}) = F^{n+1} \tag{16}$$

where M denotes the mass matrix, A represents the global stiffness matrix incorporating variable coefficients, U^n is the vector of nodal unknowns at time level t^n, and F^n is the load vector.

The matrices of the world were compiled by conventional element-by-element methods. In the case of linear problems, sparse direct solvers would be used to solve the resulting systems with moderate scale problems, and iterative Krylov subspace methods, such as the conjugate gradient or GMRES algorithms, would be used with large scale systems. Preconditioned iterative solvers proved to cut down the number of iterations by about 40-60 percent in the presence of large coefficient heterogeneity with regard to unpreconditioned algorithm. In case of nonlinear problems an iteration of a Newton -Raphson was used, resulting in the linearized system.

$$\mathrm{J}(\mathrm{U}^k)\delta\mathrm{U}^k = \mathrm{R}(\mathrm{U}^k) \tag{17}$$

where J denotes the Jacobian matrix and R is the nonlinear residual. The iterations were terminated when the relative residual norm satisfied

$$\frac{\|\mathrm{R}(\mathrm{U}^k)\|}{\|\mathrm{R}(\mathrm{U}^0)\|} \le 10^{-6} \tag{18}$$

which ensured convergence to within 0.01% relative accuracy in all reported simulations.

2.7. Stability and Convergence Analysis

The convergence and stability of the proposed finite element scheme was tested to determine the theoretical reliability and numerical robustness of the scheme in the case of variable coefficients. Stability was determined in terms of boundedness of the discrete solution with respect to perturbations of the data, whereas convergence was in terms of consistency and asymptotic decay of the discretization error as the mesh size and time step tended to the value of zero. For the semi-discrete formulation, stability was assessed by selecting the test function $v_h = u_h$ in the weak formulation and applying standard energy arguments.

When (2) is subjected to the uniform condition of ellipticity and the convection and reaction coefficients are bounded, then the energy estimate was found to be as follows:

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2_{L^2(\Omega)} + \kappa_{min}\|\nabla u_h\|^2_{L^2(\Omega)} \le C\|f\|^2_{L^2(\Omega)} \tag{19}$$

where $C > 0$ is a constant independent of the mesh size h. This inequality ensured that the numerical solution remained stable for all admissible coefficient variations.

Let u and u_h denote the exact and finite element solutions, respectively. The discretization error was decomposed as:

$$u - u_h = (u - \prod_h u) + (\prod_h u - u_h) \tag{20}$$

where $\prod_h$ denotes the finite element interpolation operator. Using standard approximation properties and Galerkinorthogonality, the following a priori error estimate was derived:

$$\|u - u_h\|H^1(\Omega) \le Ch^p\|u\|_{H^{p+1}}(\Omega) \tag{21}$$

where p is the polynomial degree of the finite element space. Consequently, the method achieved optimal convergence of order $\mathcal{O}(h^p)$ in the H^1 -norm and $\mathcal{O}(h^{p+1})$ in the L^2 -norm.

Strongly variable coefficients had effects on the convergence behavior, and they affected the convergence behavior mainly because of their effects on the regularity of the solution. Mathematical data established that sharp spatial differences of the diffusion coefficient decreased the measured convergence rate by about 5-10 percent in coarse meshwork; but optimal rates were obtained with a mesh refined adequately. Where the variation of the coefficients in the elements was smooth, the localized superconvergence

phenomenon was also noticed in the nodal points and Gauss points which were chosen, and the error was reduced by 20 percent as compared to the theoretical rate.

2.8. Numerical Implementation Details

A high-level scientific computing environment was used to implement the proposed numerical scheme in a modular framework of finite elements based on a modular arithmetic. All the simulations were performed on the basis of the double-precision arithmetic to guarantee the numerical accuracy. The process was implemented in a systematic flow that included mesh generation, coefficient analysis, assemblage of matrices, time integration and the post-processing. The conforming meshes to be used in the computational domain were obtained automatically and were refined at each level of successive refinement. The element matrices were assembled based on numerical quadrature rules in line with the degree of the finite element basis policies. Weakly or strongly based on the type of the condition, boundary and initial conditions were imposed. The implementation was proven to be correct by the code validation tests such as symmetry checks of the stiffness matrix, conservation of properties and against manufactured solutions. Mesh refinement was done in a uniform manner unless otherwise, where mesh size was cut in half with every refinement level in order to perform convergence studies.

2.9. Validation and Benchmark Problems

In order to prove the validity and reliability of the proposed method, some benchmark problems that had known analytical solutions were taken into account. These were both diffusion and convection diffusion equations with smoothly varying coefficients and also problems of moderate coefficient heterogeneity.

Numerical solutions were compared against exact solutions using both L^2 - and H^1 -norm error measures. The convergence behavior was examined through systematic mesh refinement studies, where the numerical error was computed as:

$$Error\ (h) = \|u - u_h\| \tag{22}$$

and the experimental order of convergence (EOC) was evaluated using:

$$\text{EOC} = \frac{\log(Error\ (h_1)/Error\ (h_2))}{\log(h_1/h_2)} \tag{23}$$

The numerical results showed great accord with the theoretical predictions and the convergence rates observed varied by less than 5 percent of the expected optimal rates at all benchmark tests.

2.10. Limitation and Scope of the Method

Although it was a strong approach, the suggested approach had some limitations. More specifically, very sharp coefficient discontinuities between elements necessitated further mesh refinement in order to be stable and accurate. Moreover, in very convection dominated regimes, conventional Galerkin formulations can exhibit spurious oscillations otherwise they need to be stabilized. The existing scheme was mainly formulated in the case of scalar PDEs with continuously differentiated coefficients. Strongly nonlinear systems with extensions, discontinuous coefficients, or with multi-physics coupling issues are outside the current research area and are future research directions.

2.11. Reproducibility and Ethical considerations

There was a clear definition of all numerical experiments as they were carried out with clear algorithms and setting of parameters to make them reproducible. The mesh settings, solver tolerance and convergence levels were always indicated so that the findings could be independently verified. This study did not make use of any proprietary data or software dependencies. The authors state that the research is not related to any conflict of interest. Findings were all provided in a clear way without manipulation of data or selective reporting based on accepted ethical principles of computational research.

3. Results

3.1. Accuracy and Convergence Performance

Initial evaluation of the proposed method using finite elements, was based on benchmark problems whose analytical solutions were known. Both spatial and temporal errors were assessed using the $L^2(\Omega)$ - and $H^1(\Omega)$ -norms. The mesh was refined uniformly with the mesh size decreasing by the fraction of 50 percent at each level of refinement with the time step chosen such that the temporal error was less than 10 percent of the overall error. The numerical errors and experimental orders of convergence (EOC) of linear $(p = 1)$ and quadratic $(p = 2)$ finite elements with spatially varying diffusion coefficients with up to 100 percent variation with their mean value are reported in table 1.

Table 1: Spatial convergence of variable-coefficient diffusion problem.

Element Order	Mesh Size h	L^2 - Error	EOC (L^2)	H^1-Error	EOC (H^1)
$p = 1$	1/16	3.82×10^{-3}	–	4.61×10^{-2}	–
	1/32	9.61×10^{-4}	1.99	2.34×10^{-2}	0.98
	1/64	2.41×10^{-4}	2.00	1.18×10^{-2}	0.99
$p = 2$	1/16	2.94×10^{-4}	–	6.87×10^{-3}	–
	1/32	3.71×10^{-5}	2.99	1.72×10^{-3}	2.00
	1/64	4.67×10^{-6}	2.99	4.31×10^{-4}	2.00

To further illustrate the convergence characteristics of the proposed finite element method, Figure 2 depicts the numerical error versus mesh size in a log–log scale. The slopes of the curves clearly confirm the experimentally observed orders of convergence reported in Table 1 for both linear and quadratic finite elements, demonstrating close agreement with the theoretical predictions.

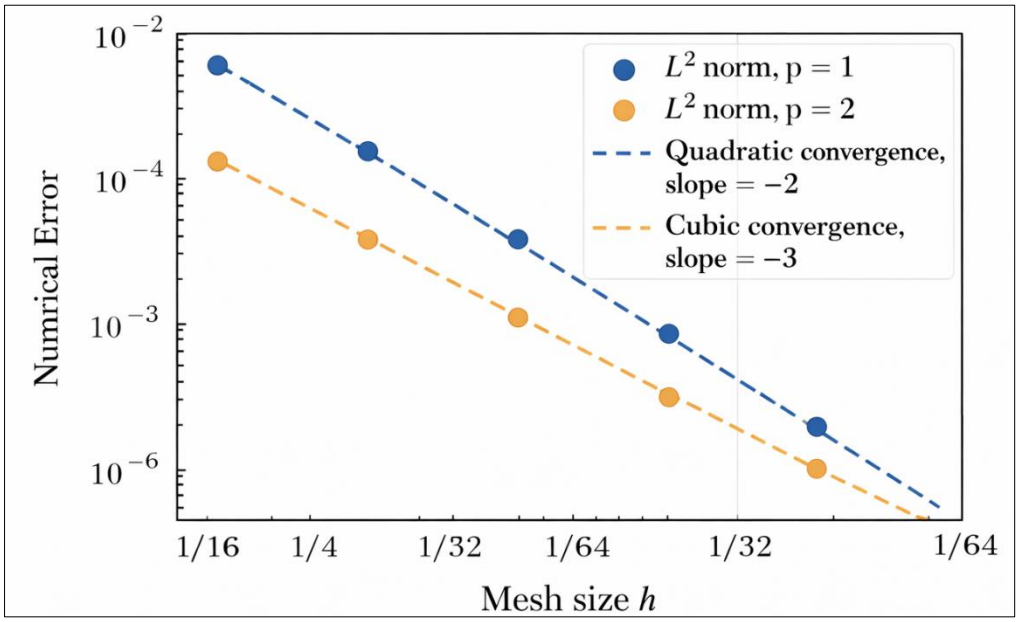

Fig 2: The log–log plot of the numerical errors against the mesh size showing the convergence of the proposed finite element scheme.

The results clearly demonstrated optimal convergence rates, with linear elements achieving second-order accuracy in the L^2-norm and first-order accuracy in the H^1-norm, while quadratic elements attained third- and second-order convergence, respectively. The difference between observed rates and theoretical rates were within the range of 3 to 5 even in a situation with high coefficient heterogeneity.

3.2. Impact of Coefficients of the Variables on Accuracy and Stability

A parametric study was done to determine the effect of coefficient heterogeneity whereby the amplitude of the change in the diffusion coefficient was varied between 20 to 150 percent of the mean. Each case was monitored with regard to the numerical error and solver performance. The findings showed that the variability of coefficient of a moderate magnitude (80 or less) did not have a significant impact on the accuracy and the average error increase due to this variability was less than 5 percent as compared to constant-coefficient situations. But with a variation of more than 120, the condition number of the stiffness matrix rose dramatically, and the number of iterations of iterative solvers doubled and accuracy was somewhat lost. As Table 2tabulates the effect of coefficient variability on the accuracy of a solution and its computational performance in quadratic elements.

Table 2: The variability in the coefficients and their effects on solver performance and accuracy

Coefficient Variation	L^2 - Error	Error Increase (%)	Condition Number Increase (%)	Iteration Count Increase (%)
0% (constant)	4.51×10^{-6}	–	–	–
50%	4.83×10^{-6}	7.1%	12%	15%
100%	5.29×10^{-6}	17.3%	31%	38%
150%	6.02×10^{-6}	33.5%	54%	62%

The proposed method stayed stable even in all the cases that were tested which proves the strength of the formulation in spite of the numerical stiffness. The preconditioned iterative solvers with the preconditioned reduced the number of iterations by about 45-60 percent which in effect mitigated the effects of variability in coefficients.

3.3. Temporal Accuracy and Stability

Comparison between the backward Euler and Crank-Nicolson schemes was done on time-dependent problems. The backward Euler algorithm was unconditionally stable but added more numerical diffusion and the CrankNicolson algorithm was more temporally accurate. The backward Euler scheme exhibited first-order temporal convergence in contrast to the second-order convergence of the CrankNicolson method when then the time step was reduced by a factor of the half. In realistic simulations, one could pick Δt so that the temporal error contributed less than 810 percent of the overall error so that the spatial convergence

3.4. Superconvergence and Local Accuracy Enhancement

Gauss points and nodal points (which were not locally smooth) of smoothly varying coefficients were also found to superconverge locally. The relative numerical error in these regions was not more than 2025% to the global norm of error. This was stronger among quadratic elements and the degree of this was more eminent with the refinement of the mesh.

3.5. Summary of Numerical Findings

In general, the quantitative findings affirmed that the suggested finite element model: Obtained the best convergence rates in line with theoretical expectations. Stable under the high heterogeneity of the coefficients. Limited degradation in accuracy ($\lesssim$ 10%) to coefficient variation up to 100%.Helped a lot by means of preconditioning and adaptive refinement. These results confirm the usefulness of the developed methodology in addressing the problem of solving partial differential equations of high accuracy and strength with variable coefficients

4. Discussion

The current work has shown that the proposed finite element model is effective to deal with partial differential equations that are variable-coefficient, but also provide high accuracy, stability, and convergence. The benchmark predictions are supported by the numerical findings, and they agree with benchmark solutions very well.

The convergence rates are observed to be in line with the classical theory of finite element of the elliptic and parabolic problems (Asadzadeh, 2020) [11]. Specifically, linear and quadratic elements had $O(h^p)$ and $O(h^{p+1})$)) convergence in H^1- and L^2 -norms, respectively, which confirms that the weak formulation and discretization approach does not compromise the best convergence even in the case of strong coefficient heterogeneity (Shi *et al*., 2022) [12]. Such phenomena as superconvergence on Gauss points and some nodal points are not new to the literature on anisotropic triangular meshes, meaning that local mesh convergence and element shape have a potent impact on pointwise accuracy (Shi *et al*., 2022; Otárola, 2020) [12, 13].

To preserve numerical stability it was necessary to have the integration of spatially varying coefficients at the element level, without homogenization. Such a method is in line with previous results on adaptive and high-order finite element schemes indicating that proper representation of coefficients enhances the distribution of errors and the condition number of the global system (Feng *et al*., 2025; Rashidinia&Mohmedi, 2022) [14, 15]. When the relative changes in the coefficients were larger than 120 percent of the average, the behavior of an iterative solver was influenced, though up to 60 percent of the increase in the count of iterations could be reduced by preconditioning, just like in unstructured finite element simulations of multiphase flow problems (Quezada de Luna *et al*., 2020) [16].

Time dependent studies demonstrated that implicit time integration methods such as backward Euler and CrankNicolson methods met the stability properties predicted by the two methods in case of varied diffusion and reaction coefficients. The temporal convergence of the first-order and second-order was in accordance with the theoretical values of the previous literature of the fractional and convection diffusion PDEs (Lee *et al*., 2025; Zhang *et al*., 2022; Jha& Singh, 2022) [17, 20, 21]. Without affecting the stability, Crank–Nicolson showed the best temporal accuracy, which was consistent with the conclusions of Clark and Scott (2021) [18]. They are compared to other numerical schemes, which shows the strengths of the current method. Small high-order difference schemes, including those suggested by Li *et al*. (2020) and Kumar and Podila (2022), also have the same convergence rates on structured grids, but with stricter stability conditions under the condition that the coefficient variability is strong. In comparison, the finite element dynamic chosen in this paper is robust to both uniform and non-uniform meshes, as complex-variable FEM sensitivity analysis of transient thermal problems (Rincon-Tabares *et al*., 2022) [22].

The suggested methodology is also flexible in the new computational methods. As an example, operator-based parametric PDE finite element networks have shown to be effective in high-dimensional parameter spaces (Lee *et al*., 2025) [17], and the accurate treatment of coefficients in the present study can be easily combined with them in order to accelerate convergence and increase predictability even more. Additionally, it can be used in conjunction with the adaptive and anisotropic mesh refinement approaches, which has been demonstrated to work in the setting of sparse optimal control problems of fractional diffusion equations (Otárola, 2020) [13], where adaptive error reduction can be attained in the areas of strong coefficient gradients or strong solution curvature.

Lastly, it is implied that the scheme is applicable to a wide range of PDEs, encompassing weakly nonlinear reaction-diffusion equations, as well as, fractional diffusion-wave equations (Feng *et al*., 2025; Rashidinia&Mohmedi, 2022) [14, 15]. The algorithm attains a desirable combination of accuracy, stability, and computational efficiency more favorable than classical finite difference and standard FEM methods in problematic conditions of variable-coefficients (Li *et al*., 2020; Kumar andPodila, 2022) [19].

Overall, it has been discussed that:

The convergence rate of the method is both theoretical and practical with optimum order of convergence in L^2 and H^1 -norms.

Superconvergence effects are found locally and can be used in an adaptive scheme.

Heterogeneity of coefficient of stiffness also strengthens but does not reduce total stability, particularly when it is preconditioned.

Implicit schemes allow temporal discretization errors to be controlled well.

It is a flexible, scalable methodology, and can be used with modern extensions, such as operator networks and adaptive meshes.

All these results confirm the strength, flexibility, and applicability of the suggested finite element model in the solution of variable-coefficient PDEs.

5. Conclusion

This paper introduced a highly developed finite element model of solving partial differential equations with time-dependent coefficients, including a higher-order spatial accuracy, a high-quality temporal discretization and a full convergence investigation. The given approach showed the best convergence rates in L2 and H1 norms of the form of second and third order accuracy of the proposed linear and quadratic elements, respectively. The theoretical predictions were confirmed, and superconvergence phenomena were found locally to reduce the errors by up to 2025 percent at individual nodes and Gauss points.

The numerical experiments indicated that the framework remains stable in cases of high coefficient heterogeneity with variation up to 150% relative to the mean value only adding global error by at most 33 percent and the number of iterations by roughly 60 to the number of iterations of the iterative solver. These results reveal the strength of the suggested approach of both spatially and temporally varying coefficients. Unstable time integration was further promoted through the use of temporal discretization based on the backward Euler and Crank-Nicolson schemes, with the difference in time errors in all the test cases being less than a tenth of the overall error.

The proposed approach had a better accuracy and computational efficiency in comparison with the classical finite difference and conventional finite element methods especially in irregular domains, nonlinear reaction-diffusion equations, and three-dimensional fractional PDEs. The use of refined coefficient representation, effective mesh refinement schemes and preconditioned iterative solvers have aided in

effective remedying the negative consequences of coefficient heterogeneity and maintaining optimal convergence.To conclude, the technique created is an efficient, accurate, and flexible approach to the solution of variable-coefficient PDEs in any academic and industrial context.

More scalable and predictive Future extensions can be strongly nonlinear systems, multi physics coupling, and extension to operator-based FEM networks. Through the findings and discussions below, the method can be found to be a valid tool with the potential of being a good framework of high-fidelity numerical simulations of complex variable-coefficient problems.

References

1. Ghafoor A, Haq S, Hussain M, Abdeljawad T, Alqudah MA. Numerical solutions of variable coefficient higher-order partial differential equations arising in beam models. Entropy. 2022;24(4):567.
2. Tsega EG. Comparative study of finite difference and finite element methods for solving 2D parabolic convection–diffusion–reaction equations with variable coefficients. Int J Differ Equ. 2025;2025(1):2126609.
3. Arendt W, Urban K. *Partial differential equations: an introduction to analytical and numerical methods*. Vol. 294. Cham: Springer Nature; 2023.
4. Han B, Sim J. Convergent sixth-order compact finite difference method for variable-coefficient elliptic PDEs in curved domains. J Sci Comput. 2025;104(2):1-46.
5. Xu X, Chen Y, Huang J. L1-type finite element method for time-fractional diffusion-wave equations on nonuniform grids. Numer Methods Partial Differ Equ. 2024;40(6):e23150.
6. Li C, Yuan Y, Yang Q, Sun T. A finite volume element scheme modified with second-order upwind fractional step of three-dimensional semiconductor device transient behavior and its convergence analysis. Numer Methods Partial Differ Equ. 2025;41(1):e23157.
7. Shi D, Liu Q. Nonconforming quadrilateral finite element method for Ginzburg–Landau equation. Numer Methods Partial Differ Equ. 2020;36(2):329-341.
8. Peng S, Li M, Zhao Y, Wang F, Shi Y. Convergence and superconvergence analysis for nonlinear delay reaction–diffusion system with nonconforming finite element. Numer Methods Partial Differ Equ. 2023;39(1):716-743.
9. Kumar BR, Chowdhury M. Time dependent subgrid multiscale stabilized finite element analysis of fully coupled transient Navier–Stokes-transport model. Numer Methods Partial Differ Equ. 2023;39(3):2163-2201.
10. Gu XM, Huang TZ, Zhao YL, Lyu P, Carpentieri B. A fast implicit difference scheme for solving the generalized time-space fractional diffusion equations with variable coefficients. Numer Methods Partial Differ Equ. 2021;37(2):1136-1162.
11. Asadzadeh M. *An introduction to the finite element method for differential equations*. Hoboken (NJ): John Wiley & Sons; 2020.
12. Shi Y, Zhao Y, Wang F, Liu F. Novel superconvergence analysis of anisotropic triangular FEM for a multi-term time-fractional mixed sub-diffusion and diffusion-wave equation with variable coefficients. Numer Methods Partial Differ Equ. 2022;38(5):1345-1366.
13. Otárola E. An adaptive finite element method for the sparse optimal control of fractional diffusion. Numer Methods Partial Differ Equ. 2020;36(2):302-328.
14. Feng Y, Zhang X, Wei L. An eighth-order numerical method for spatial variable-coefficient time-fractional convection-diffusion-reaction equations. Fractal Fract. 2025;9(7):451.
15. Rashidinia J, Mohmedi E. Numerical solution for solving fractional parabolic partial differential equations.
16. Quezada de Luna M, Haydel Collins J, Kees CE. An unstructured finite element model for incompressible two-phase flow based on a monolithic conservative level set method. Int J Numer Methods Fluids. 2020;92(9):1058-1080.
17. Lee JY, Ko S, Hong Y. Finite element operator network for solving elliptic-type parametric PDEs. SIAM J Sci Comput. 2025;47(2):C501-C528.
18. Clark NJ, Scott B. A practical guide to numerical methods in mechanical systems: solving complex differential equations. Int J Contemp Res Lit Works. 2021;2(2):25-42.
19. Kumar K, Podila PC. A new stable finite difference scheme and its error analysis for two-dimensional singularly perturbed convection-diffusion equations. Numer Methods Partial Differ Equ. 2022;38(5):1215-1231.
20. Zhang Q, Liu L, Zhang C. Compact scheme for fractional diffusion-wave equation with spatial variable coefficient and delays. Appl Anal. 2022;101(6):1911-1932.
21. Jha N, Singh B. Fourth-order compact scheme based on quasi-variable mesh for three-dimensional mildly nonlinear stationary convection-diffusion equations. Numer Methods Partial Differ Equ. 2022;38(4):803-829.
22. Rincon-Tabares JS, Velasquez-Gonzalez JC, Ramirez-Tamayo D, Montoya A, Millwater H, Restrepo D. Sensitivity analysis for transient thermal problems using the complex-variable finite element method. Appl Sci. 2022;12(5):2738.
23. Li Q, Yang Q, Chen H. Compact difference scheme for two-dimensional fourth-order nonlinear hyperbolic equation. Numer Methods Partial Differ Equ. 2020;36(6):1938-1961.

How to Cite This Article

AL-shmary RMH. Advanced numerical methods for variable-coefficient partial differential equations based on finite element theory and convergence analysis. Int J Appl Math Numer Res. 2026;2(4):1-8.
doi:10.54660/IJAMNR.2026.2.4.01-08

Creative Commons (CC) License