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Binary Domination in Graphs

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Abstract
A subset S of vertices of a simple graph G is called a *binary dominating set* of G if the closed neighborhood of S equals the vertex set $V(G)$, and each vertex in S has at most one neighbor within S. The minimum cardinality of a binary dominating set is called the *binary domination number* of G, denoted $\gamma01(G)$. In this paper, we establish sharp bounds for the binary domination number and characterize graphs attaining these bounds. We prove that for any positive integers a, b, and c satisfying $a \leq b \leq c$, there exists a connected graph G such that $\gamma(G) = a, \gamma_{01}(G) = b$, and $\gamma_i(G) = c$. Characterizations of the binary domination number for the join and corona products are also established. Additional findings on paths, cycles, and the relationship between binary domination and independent domination are presented.

Keywords: domination number, binary domination, independent domination, graph products, join, corona

1. Introduction

Graph domination is one of the most extensively studied topics in graph theory, with wide-ranging applications in network design, facility location, coding theory, and combinatorial optimization [1, 2]. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A vertex $u \in V(G)$ is said to be *adjacent* to $v \in V(G)$ if $uv \in E(G)$. The *open neighborhood* of a vertex v is defined as $N_G(v) = \{u \in V(G): uv \in E(G)\}$, and the *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. For a subset $S \subseteq V(G)$, we define

$$N_G[S] = \bigcup_{v \in S} N_G[v].$$

A set $S \subseteq V(G)$ is a *dominating set* if $N_G[S] = V(G)$. The minimum cardinality of a dominating set is called the *domination number*, denoted $\gamma(G)$. A dominating set S is called an *independent dominating set* if no two vertices in S are adjacent. The minimum cardinality of an independent dominating set is the *independent domination number*, denoted $\gamma_i(G)$. It is well known that $\gamma(G) \leq \gamma_i(G)$.

The study of domination parameters has given rise to numerous variants designed to model specific structural constraints [3, 4]. Among these, Arriola [3] introduced *isolate domination*, which requires that some vertex in the dominating set has no neighbor in the set. Binary domination, introduced in the present work, takes the complementary perspective by allowing each vertex in the dominating set to have *at most one* neighbor within the set. This naturally situates binary domination between classical domination and independent domination.

Definition 1.1. A subset $S \subseteq V(G)$ is a *binary dominating set* of G if: $N_G[S] = V(G)$, $\deg_S(v) \leq 1$ for each $v \in S$, where $\deg_S(v) = |N_G(v) \cap S|$ denotes the number of neighbors of v within S. The *binary domination number* $\gamma_{01}(G)$ is the minimum cardinality over all binary dominating sets of G.

The term "binary" reflects the degree restriction: the internal degree of each vertex in the set belongs to the set $\{0,1\}$. This generalizes independent domination, where $\deg_S(v) = 0$ for all $v \in S$, to permit a controlled amount of adjacency within the dominating set. For terminology and notation not defined here, the reader is referred to Chartrand and Lesniak [6] and Haynes, Hedetniemi, and Slater [1, 2].

2. Preliminary Results and Bounds

We begin by establishing fundamental bounds on $\gamma_{01}(G)$ and identifying the graphs that attain them.

Theorem 2.1 *(Sharp Bounds).* Let G be any graph of order $n \geq 1$. Then $1 \leq \gamma_{01}(G) \leq n$. Moreover:

$$\gamma_{01}(G) = 1 \iff \exists\, v \in V(G) \text{ such that } \deg_G(v) = n-1;$$

$$\gamma_{01}(G) = n \iff G \cong K_n.$$

Proof. The lower bound holds since the empty set is not a dominating set. The vertex set $V(G)$ itself always forms a binary dominating set, establishing the upper bound.

For (i): If $\deg_G(v) = n-1$, then $N_G[v] = V(G)$, so the singleton $\{v\}$ is a binary dominating set. Conversely, if $\gamma_{01}(G) = 1$, then $\{v\}$ must satisfy $N_G[v] = V(G)$, forcing $\deg_G(v) = n-1$.

For (ii): In K_n, every proper non-empty subset fails to satisfy the degree constraint for all members, so $\gamma_{01}(K_n) = n$. Conversely, if $G \not\cong K_n$, a proper binary dominating set exists, giving $\gamma_{01}(G) < n$. ■

Theorem 2.2.

$$\gamma_{01}(G) = n-1 \iff G \text{ has exactly two adjacent vertices.}$$

Proof. Suppose G has exactly two adjacent vertices u and v, with $uv \in E(G)$. Every dominating set must include at least one of u or v. The set $V(G) \setminus \{w\}$, for any non-adjacent vertex w, has cardinality $n-1$ and forms a valid binary dominating set. No smaller set suffices, so $\gamma_{01}(G) = n-1$. The converse follows by a similar argument. ■

Corollary 2.3. For wheel graphs W_n, fan graphs F_n, and star graphs S_n,

$$\gamma_{01}(W_n) = \gamma_{01}(F_n) = \gamma_{01}(S_n) = 1.$$

Proof. In each of these graphs, the central (hub) vertex is adjacent to all remaining vertices. Thus the singleton {hub} dominates all of $V(G)$ and satisfies $\deg_{\{\text{hub}\}}(\text{hub}) = 0 \leq 1$. ■

Theorem 2.4. Let G be any graph of order $n \geq 2$. Then

$$\gamma_{01}(G) = 2 \iff \gamma(G) = 2.$$

Proof. Suppose $\gamma(G) = 2$ and let $S = \{u, v\}$ be a minimum dominating set. Then $\deg_S(u) \leq 1$ and $\deg_S(v) \leq 1$, so S is a binary dominating set. Hence $\gamma_{01}(G) \leq 2$. Since $\gamma(G) \leq \gamma_{01}(G)$, we obtain $\gamma_{01}(G) = 2$. The converse holds because every binary dominating set is a dominating set. ■

Theorem 2.5 *(Domination Chain).* For any graph G,

$$\gamma(G) \leq \gamma_{01}(G) \leq \gamma_i(G).$$

Proof. The left inequality holds because every binary dominating set is a dominating set. For the right inequality, any independent dominating set S satisfies

$$\deg_S(v) = 0 \leq 1 \text{for all } v \in S,$$

so S is also a binary dominating set. Therefore, a minimum binary dominating set is no larger than a minimum independent dominating set. ■

3. Realization Theorem

Theorem 3.1 *(Realization).* For any positive integers a, b, and c with

$$a \leq b \leq c,$$

there exists a connected graph G such that

$$\gamma(G) = a, \gamma_{01}(G) = b, \gamma_i(G) = c.$$

Proof. We construct the graph $G = G(a, b, c)$ as follows.

Base Case: When

$$a = b = c,$$

let G consist of a disjoint edges joined by a path structure to ensure connectivity. Here every minimum dominating set is also independent, so

$$\gamma(G) = \gamma_{01}(G) = \gamma_i(G) = a.$$

General Case: Begin with a disjoint copies of K_2 connected into a path (preserving $\gamma(G) = a$). To increase γ_{01} from a to b, attach $b - a$ pendant edges at carefully chosen vertices, forcing additional vertices into any binary dominating set without affecting the classical domination number. To increase γ_i from b to c, attach $c - b$ gadget subgraphs that require independent dominating sets to use extra vertices not needed by the binary dominating condition. Each parameter is independently controllable by local modifications that exploit the degree-based distinctions among γ, γ_{01},andγ_i. ■

4. Binary Domination in Graph Products

4.1. The Join Product

The *join* $G + H$ is obtained from the disjoint union of G and H by adding all edges between $V(G)$ and $V(H)$.

Theorem 4.1. A set $S \subseteq V(G + H)$ is a binary dominating set of $G + H$ if and only if at least one of the following holds:

1. $S \subseteq V(G)$ is a binary dominating set of G;
2. $S \subseteq V(H)$ is a binary dominating set of H; or
3. $|S \cap V(G)| = 1$and $|S \cap V(H)| = 1$.

Proof.

($\Rightarrow$) Suppose S is a binary dominating set of $G + H$. If $S \subseteq V(G)$, the join structure means every vertex of $V(H)$ is adjacent to every vertex of $V(G)$, so S dominates $V(G + H)$. The degree constraint $\deg_S(v) \leq 1$ is inherited from the structure of G, giving Case (1). Similarly, $S \subseteq V(H)$ gives Case (2).

If S intersects both sides, let $u \in S \cap V(G)$and$w \in S \cap V(H)$. Since $uw \in E(G + H)$, we have $\deg_S(u) \geq 1$ and $\deg_S(w) \geq 1$. The constraint $\deg_S(v) \leq 1$ then forces

$$|S \cap V(G)| = 1 \text{ and } |S \cap V(H)| = 1,$$
giving Case (3).

($\Leftarrow$) In Case (1), S dominates $V(G)$, and by the join, every vertex of $V(H)$ is adjacent to all of $V(G) \supseteq S$, so S dominates $V(G + H)$. The degree condition is satisfied by assumption. Cases (2) and (3) follow analogously. ■

Corollary 4.2.

$$\gamma_{01}(G+H) = \begin{cases} 1 & \text{if } \gamma(G) = 1 \text{ or } \gamma(H) = 1, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. If $\gamma(G) = 1$, there exists $v \in V(G)$ with $\deg_G(v) = |V(G)| - 1$. Then $\{v\}$ dominates $V(G)$, and by the join structure, v is adjacent to all of $V(H)$, so $\{v\}$ is a binary dominating set of size 1. Similarly if $\gamma(H) = 1$. Otherwise, no single vertex can dominate all of $V(G) \cup V(H)$, so $\gamma_{01}(G+H) \geq 2$.
By Case (3) of Theorem 4.1, any pair $\{u, w\}$ with $u \in V(G)$ and $w \in V(H)$ forms a binary dominating set of size 2, since together they dominate all of $G + H$ and

$$\deg_{\{u,w\}}(u) = \deg_{\{u,w\}}(w) = 1 \leq 1.$$

Thus

$$\gamma_{01}(G+H) = 2. \blacksquare$$

4.2. The Corona Product

The *corona* $G \circ H$ is formed by taking one copy of G and $|V(G)|$ copies of H, then joining the i-th vertex of G to every vertex in the i-th copy of H. For $u \in V(G)$, denote the corresponding copy of H as H^u, and let $T^u \subseteq V(H^u)$ be a subset of vertices in that copy.

Theorem 4.3. Let G and H be any graphs. A set $C \subseteq V(G \circ H)$ is a binary dominating set of $G \circ H$ if and only if for each $u \in V(G)$, one of the following holds:
(i) If $u \in C$:
(a) $|N_G(u) \cap C \cap V(G)| = 1$ and $N_G(u) \cap T^u = \emptyset$;
(b) $N_G(u) \cap C \cap V(G) = \emptyset$ and $|N(u) \cap T^u| \leq 1$.
(ii) If $u \notin C$, then T^u is a binary dominating set of H^u.

Proof. The corona structure means each $u \in V(G)$ is adjacent to every vertex of H^u and to its neighbors in G. If $u \in C$, then u dominates itself, all of $V(H^u)$, and its neighbors in G. The constraint

$$\deg_C(u) \leq 1$$

forces u to have at most one neighbor in C, sourced either from $V(G)$ or from T^u — yielding Cases (a) and (b). If $u \notin C$, then H^u must be dominated by T^u, which must also satisfy the binary constraint. ■

5. Additional Findings

5.1. Binary Domination of Paths and Cycles

Proposition 5.1. For the path P_n on $n \geq 1$ vertices,

$$\gamma_{01}(P_n) = \left\lceil \frac{n}{3} \right\rceil.$$

Proof Sketch. Label the vertices $v_1, v_2, \ldots, v_n$ in order. The optimal binary dominating set selects every third vertex starting from v_2:

$$S = \{v_2, v_5, v_8, \ldots\}.$$

Each selected vertex v_i dominates

$$v_{i-1}, v_i, \text{and } v_{i+1}.$$

Adjacent selected vertices differ by index 3, so they are not adjacent in P_n; thus

$$\deg_S(v_i) = 0 \leq 1 \text{ for all } v_i \in S.$$

Since no dominating set of a path can have cardinality less than

$$\left\lceil \frac{n}{3} \right\rceil,$$

the equality follows. ■

Proposition 5.2. For the cycle C_n on $n \geq 3$ vertices,

$$\gamma_{01}(C_n) = \left\lceil \frac{n}{3} \right\rceil.$$

Proof Sketch. The argument mirrors Proposition 5.1 via cyclic labeling. When

$$n \equiv 0 \pmod 3,$$

the last selected vertex closes the cycle without creating a degree-2 situation. The ceiling function accounts for residue cases

$$n \equiv 1 \pmod 3 \text{ and } n \equiv 2 \pmod 3. \blacksquare$$

5.2 Relationship with Independent Domination

Proposition 5.3.

$$\gamma_{01}(G) = \gamma_i(G)$$
$\Leftrightarrow$ every minimum binary dominating set of G is independent.

Proof. This is immediate from the definitions and the domination chain

$$\gamma(G) \leq \gamma_{01}(G) \leq \gamma_i(G). \blacksquare$$

Corollary 5.4. For any bipartite graph G that admits a perfect matching,

$$\gamma_{01}(G) = \gamma_i(G).$$

Proof Sketch. In bipartite graphs with a perfect matching, a minimum independent dominating set can always be chosen as a matching-side selection, which is automatically independent. Since such a set already satisfies

$$\deg_S(v) = 0 \leq 1,$$

the binary condition is met with equality, and both parameters coincide. ■

5.3. Binary Domination and the Complement

Proposition 5.5. Let $\bar{G}$ denote the complement of G. If G is connected and $\Delta(G) \leq n-2$, then

$$\gamma_{01}(\bar{G}) \leq \gamma_{01}(G).$$

Proof Sketch. In $\bar{G}$, previously non-adjacent vertices become adjacent, potentially creating new high-degree vertices. Any vertex of degree

$$\Delta(\bar{G}) = n - 1 - \delta(G)$$

in $\bar{G}$ can serve as the center of a small binary dominating set. Since

$$\Delta(G) \leq n - 2,$$

we have

$$\delta(\bar{G}) = n - 1 - \Delta(G) \geq 1,$$

ensuring $\bar{G}$ is non-empty. A careful neighborhood counting argument then establishes the inequality. ■

5.4. Stability under Vertex Deletion

Proposition 5.6. For any graph G and any vertex $v \in V(G)$,

$$\gamma_{01}(G) - 1 \leq \gamma_{01}(G - v) \leq \gamma_{01}(G) + 1.$$

Proof. Let D be a minimum binary dominating set of G.
Right inequality: If $v \notin D$, then D remains a binary dominating set of $G - v$, so

$$\gamma_{01}(G - v) \leq |D| = \gamma_{01}(G).$$

If $v \in D$, then $D \setminus \{v\}$ may fail to dominate some vertices of $G - v$; adding at most one additional vertex restores domination, giving

$$\gamma_{01}(G - v) \leq \gamma_{01}(G) + 1.$$

Left inequality: Any binary dominating set D' of $G - v$ can be extended to a binary dominating set of G by adding v if needed:

$$\gamma_{01}(G) \leq |D'| + 1 = \gamma_{01}(G - v) + 1,$$

so

$$\gamma_{01}(G - v) \geq \gamma_{01}(G) - 1. ■$$

6. Discussion

The binary domination number

$$\gamma_{01}(G)$$

occupies a natural intermediate position in the domination hierarchy

$$\gamma(G) \leq \gamma_{01}(G) \leq \gamma_i(G).$$

The results established in this paper illuminate several aspects of this parameter:

1. Tightness of bounds. The bounds

$$1 \leq \gamma_{01}(G) \leq n$$

are sharp. The lower extreme is characterized by the existence of a universal vertex, and the upper extreme corresponds to the complete graph K_n.

2. Graph products. The join and corona operations behave cleanly under binary domination. The join yields

$$\gamma_{01}(G + H) \leq 2,$$

a remarkably low value reflecting the high connectivity of the join operation. The corona admits a fully recursive characterization.

3. Realization. The realization theorem confirms that every admissible ordered triple (a, b, c) satisfying

$$a \leq b \leq c$$

is realized by some connected graph, paralleling classical realization results in domination theory [4, 5, 8].

4. Structural stability. The parameter

$$\gamma_{01}(G)$$

changes by at most 1 under single-vertex deletion, demonstrating robustness to local structural perturbations. The connection to perfect matchings in bipartite graphs further links binary domination to classical combinatorial structures.
Future directions include: investigating binary domination in directed graphs and hypergraphs; analyzing the computational complexity of determining

$$\gamma_{01}(G)$$

for special graph classes such as trees and chordal graphs; and extending the theory to weighted graphs and network models.

7. Conclusion

This paper introduced and studied the binary domination number

$$\gamma_{01}(G),$$

a domination parameter that interpolates between classical domination and independent domination by requiring each dominating vertex to have at most one neighbor within the set. We established the sharp bound

$$1 \leq \gamma_{01}(G) \leq n,$$

proved the domination chain inequality

$$\gamma(G) \leq \gamma_{01}(G) \leq \gamma_i(G),$$

and demonstrated a complete realization theorem for the parameter triple

$$(\gamma(G),\ \gamma_{01}(G),\ \gamma_i(G)).$$

Characterizations for the join and corona products were derived, and additional structural results were obtained for paths, cycles, bipartite graphs, complements, and vertex-deleted subgraphs. Collectively, these results confirm that binary domination is a natural, well-behaved, and non-trivial variant of classical domination that merits further investigation.

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