International Journal of Applied Mathematics and Numerical Research

Optimal Investment and Consumption under Inflation and Labour Income Uncertainty

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Article Info

Impact Factor (RSIF): 8.62
ISSN (Online): 3107-7110
Volume: 02
Issue: 04
Received: 07-05-2026
Accepted: 09-06-2026
Published: 11-07-2026
Page No: 36-49

Abstract
This paper develops a finite-horizon stochastic-control model for the joint determination of household consumption, saving and portfolio allocation when both the price level and labour income are uncertain. The formulation begins in nominal units and uses Itô's formula to obtain internally consistent real returns and real earnings. Consequently, inflation affects the household not only through a reduction in purchasing power, but also through asset inflation covariances and the covariance between nominal wage growth and the price level. The household invests in a real locally riskless account and multiple risky assets, receives stochastic labour income, consumes in real units and values a terminal bequest under constant relative risk aversion preferences. A dynamic programming principle yields a Hamilton Jacobi Bellman equation in financial wealth, real income and time. In a dynamically complete benchmark, labour income is capitalised as risk adjusted human wealth. The problem then reduces to a Merton problem in total wealth, producing closed-form consumption and portfolio rules. The optimal financial portfolio contains a speculative demand and a human capital hedge; its sign and magnitude depend on the correlation of income innovations with traded returns. For incomplete markets, borrowing limits and portfolio constraints, the paper gives a reduced nonlinear HJB equation, boundary conditions and a monotone policy-iteration scheme. Analytic comparative statics show how risk aversion, inflation exposure, income volatility, retirement horizon and market completeness alter saving and risky investment. A transparent illustrative calibration is used to verify the closed-form formulas and to specify reproducible numerical experiments without presenting synthetic calculations as observed household evidence. The framework provides a tractable bridge between financial mathematics, actuarial life-cycle analysis and household policy design.

DOI: https://doi.org/10.54660/IJAMNR.2026.2.4.36-49

Keywords: stochastic optimal control, consumption, portfolio choice, inflation risk, labour income
JEL classification: C61; D14; D15; E21; G11

Introduction

Households make consumption and investment decisions in nominal markets but derive welfare from real consumption. This apparently simple distinction is mathematically important. A nominal deposit may be certain in currency units and risky in purchasing-power units, while a nominal wage may rise even when real earnings fall. Moreover, labour income is usually non-tradable, only partly insurable and correlated with aggregate asset returns. A model that adds an inflation rate to an otherwise real wealth equation can therefore miss the Itô correction and the covariance terms that govern actual purchasing-power risk.
The continuous-time portfolio problem was established by Merton [1, 2], building on the intertemporal insight of Samuelson [3]. In the benchmark model, an investor with constant-relative-risk-aversion (CRRA) preferences consumes a constant horizon-

dependent fraction of wealth and holds a risky share proportional to the market price of risk and inversely proportional to risk aversion. Labour income changes this conclusion because the present value of future earnings is an implicit asset. Bodie, Merton and Samuelson [4] showed that labour flexibility and human capital materially affect life-cycle allocation. Subsequent work demonstrated that non-tradable income risk modifies both precautionary saving and risky investment [5, 6], while inflation creates distinct intertemporal hedging demands [7, 8].

This study integrates these mechanisms in one mathematically explicit framework. Its first objective is to derive real income and real asset returns from nominal primitives. Its second is to characterise optimal consumption, saving and portfolio allocation by dynamic programming. Its third is to separate the complete-market solution into total-wealth demand and a hedge against human-capital risk. Its fourth is to formulate the incomplete-market problem in a form suitable for monotone numerical approximation. The final objective is to obtain interpretable comparative statics relevant to household finance, pension design and inflation protection.

The principal contribution is not the claim that every labour-income shock is tradable. Rather, market completeness is used as an analytic benchmark, after which the unspanned component is restored explicitly. This separation makes clear which conclusions are exact and which require numerical solution. The paper also distinguishes an illustrative calibration from empirical estimation. All numerical values are reproducible model inputs; no simulated quantity is described as evidence about a particular population.

The remainder of the paper is organised as follows. Section 2 positions the model in the literature. Section 3 defines the probability space and derives real returns and real earnings. Section 4 states the household problem and admissibility conditions. Section 5 derives the HJB equation. Section 6 solves the complete-market benchmark. Sections 7 and 8 address unspanned income risk and constraints. Sections 10–11 provide the computational design and comparative statics. The last sections discuss actuarial implications, limitations and conclusions.

Related Literature and Research Gap

Three strands of research motivate the analysis. The first is the classical consumption–portfolio problem. Under diffusion returns and time-additive utility, stochastic dynamic programming converts an intertemporal choice into a nonlinear partial differential equation. The homogeneity of CRRA utility then permits an analytic reduction. Rigorous martingale and duality treatments of consumption–investment problems were developed by Karatzas *et al.* [9] and Cox and Huang [10]; comprehensive accounts are given by Karatzas and Shreve [11] and Pham [12].

The second strand introduces non-traded human capital. Viceira [5] shows that risky labour income and retirement alter the optimal risky share. Cocco, Gomes and Maenhout [6] solve a realistic life-cycle problem with mortality and borrowing restrictions. Campbell and Viceira [13] interpret long-horizon demands as the sum of myopic and intertemporal hedging components. Gomes, Kotlikoff and Viceira [14] further show that flexible labour supply can materially change welfare and the allocation generated by life-cycle funds. The recurring lesson is that financial wealth alone is an incomplete measure of risk-bearing capacity.

The third strand models purchasing-power risk. Brennan and Xia [7] derive dynamic allocation rules when only nominal assets are available and inflation is uncertain. Munk, Sørensen and Vinther [8] combine inflation uncertainty, mean-reverting returns and stochastic interest rates. In actuarial applications, inflation-linked assets also matter for defined-contribution pension wealth and replacement-rate risk [15, 16]. These studies establish that nominal certainty is not real certainty and that horizon-dependent inflation hedging can be large.

The present paper fills a modelling gap between these strands. Many tractable household models specify labour income directly in real terms, obscuring its joint dependence on nominal wage shocks and inflation. Conversely, rich inflation models often omit consumption or non-traded earnings. Here, both real income and real asset returns are derived from common nominal primitives, so all covariance adjustments are visible. The resulting complete-market rule has a transparent human-capital hedge, whereas the incomplete-market model retains a low-dimensional state space appropriate for computation.

Nominal Economy and Real State Variables

We begin by specifying the probabilistic setting on which the nominal and real state processes are constructed.

Let

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$$

be a complete filtered probability space with filtration

$$\mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$$

satisfying the usual conditions. It supports an m-dimensional standard Brownian motion

$$W = (W^1, \ldots, W^m)^\top$$

All processes are progressively measurable with respect to $\mathbb{F}$. The finite date T is interpreted as retirement, death, annuitisation or the terminal planning horizon.

Vectors are columns. For

$$u, v \in \mathbb{R}^m, u^\top v$$

denotes their inner product and

$$\|u\| = (u^\top u)^{1/2}$$

Coefficients are initially constant to expose the economic mechanisms. Section 14 discusses stochastic coefficients and regime switching.

Within this setting, the price level and nominal wage process are modelled jointly so that real labour income follows from their ratio rather than from an independently imposed equation.

Let $P_t > 0$ denote the price index and $L_t > 0$ nominal labour income per unit time. Their dynamics are

$$\frac{\mathrm{d}P_t}{P_t} = \pi_P\,\mathrm{d}t + \sigma_P^\top\,\mathrm{d}W_t,$$
$$\frac{\mathrm{d}L_t}{L_t} = \mu_L\,\mathrm{d}t + \sigma_L^\top\,\mathrm{d}W_t,$$

where π_P is expected price-level growth, and $\sigma_P, \sigma_L \in \mathbb{R}^m$ encode inflation and wage innovations. The formulation permits both aggregate and idiosyncratic wage shocks by enlarging W.
Real labour income is

$$Y_t = \frac{L_t}{P_t}$$

Applying Itô's formula to the ratio gives

$$\begin{aligned}\frac{\mathrm{d}Y_t}{Y_t} &= \alpha_Y\,\mathrm{d}t + \eta^\top\,\mathrm{d}W_t,\\ \alpha_Y &= \mu_L - \pi_P + \|\sigma_P\|^2 - \sigma_L^\top\sigma_P,\\ \eta &= \sigma_L - \sigma_P.\end{aligned}$$

Thus expected real wage growth is not simply $\mu_L - \pi_P$. The quadratic variation of $\frac{1}{P_t}$ and the wage–inflation covariance contribute to the drift. If wage innovations perfectly match price innovations, then $\eta = 0$ and purchasing-power earnings are locally deterministic, although nominal earnings remain random.
The same nominal-to-real transformation is now applied to traded securities, thereby preserving the covariance corrections generated by inflation.
Consider n risky nominal securities with prices S_t^i:

$$\frac{\mathrm{d}S_t^i}{S_t^i} = \mu_i^N\,\mathrm{d}t + (\sigma_i^N)^\top\,\mathrm{d}W_t, \qquad i = 1, \dots, n.$$

For the real price

$$\tilde{S}_t^i = \frac{S_t^i}{P_t},$$

another application of Itô's formula yields

$$\begin{aligned}\frac{\mathrm{d}\tilde{S}_t^i}{\tilde{S}_t^i} &= \mu_i^R\,\mathrm{d}t + \sigma_i^\top\,\mathrm{d}W_t,\\ \mu_i^R &= \mu_i^N - \pi_P + \|\sigma_P\|^2 - (\sigma_i^N)^\top\sigma_P,\\ \sigma_i &= \sigma_i^N - \sigma_P.\end{aligned}$$

These equations display the two channels of inflation: an expected erosion of real return and a covariance correction. An asset whose nominal payoff rises with the price index can have small real volatility even if its nominal volatility is large.
For analytic clarity, suppose that a locally riskless real account B^R is available,

$$\frac{\mathrm{d}B_t^R}{B_t^R} = r\,\mathrm{d}t,$$

where r is the real short rate. This security may be interpreted as an inflation-indexed account. The case in which only a nominal money-market account is available is recovered by including its real value among the risky assets; its real volatility is $-\sigma_P$.
Define the real excess-return vector

$$b = (\mu_1^R - r, \dots, \mu_n^R - r)^\top$$

and the $n \times m$ volatility matrix Σ whose ith row is $\sigma_i^\top$. The instantaneous risky-return covariance matrix is

$$C = \Sigma\Sigma^\top.$$

Unless stated otherwise, C is positive definite. In a complete Brownian market $n = m$ and Σ is nonsingular; the market price of risk is

$$\lambda = \Sigma^{-1}b.$$

The resulting covariance structure has a direct economic interpretation.
The vector η measures real-income exposure, whereas each row of Σ measures an asset's real-return exposure. Their covariance is $\Sigma\eta$. A positive element means that the corresponding asset tends to pay well when real labour income is high; it is therefore a poor hedge against earnings risk. A negative element makes the asset a natural human-capital hedge. Inflation can reverse this classification because both vectors subtract σ_P. Hence an apparently weak nominal wage–equity correlation does not imply weak correlation between real wages and real equity returns.

Household Optimisation Problem

Given the real return system above, the household's real financial wealth can be written as a controlled diffusion.
Let X_t be real financial wealth, $c_t \geq 0$ the real consumption rate and

$$a_t = (a_t^1, \dots, a_t^n)^\top$$

the vector of real currency amounts invested in risky assets. The remainder

$$X_t - \mathbf{1}^\top a_t$$

is invested in the real account. Before retirement, wealth satisfies

$$\mathrm{d}X_t = (rX_t + a_t^\top b + Y_t - c_t)\,\mathrm{d}t + a_t^\top\Sigma\,\mathrm{d}W_t.$$

The instantaneous real saving flow is

$$s_t = Y_t + rX_t + a_t^\top b - c_t,$$

and the saving rate relative to labour income is s_t/Y_t. Thus optimal saving is an implication of optimal consumption and portfolio choice, rather than an independent control.
The household evaluates admissible consumption and investment policies through time-additive expected utility and a terminal bequest.
For risk aversion

$$\gamma > 0, \gamma \neq 1,$$

define CRRA utility

$$U(q) = \frac{q^{1-\gamma}}{1-\gamma}, \qquad q > 0.$$

The logarithmic case is obtained by continuity as $\gamma \to 1$. Let $\delta > 0$ be subjective discounting and $\beta > 0$ the terminal bequest weight. At state (t, x, y) the household maximises

$$J_{t,x,y}(c,a) = \mathbb{E}_{t,x,y}\left[\int_t^T e^{-\delta(s-t)}\, U(c_s)\,\mathrm{d}s + e^{-\delta(T-t)}\beta U(X_T)\right].$$

The value function is

$$V(t,x,y) = \sup_{(c,a)\in\mathcal{A}(t,x,y)} J_{t,x,y}(c,a).$$

Assumption 1 (Admissibility). An admissible control (c,a) is progressively measurable, satisfies $c_t > 0$ almost everywhere,

$$\mathbb{E}\int_0^T \left(c_t + \|a_t\|^2\right)\mathrm{d}t < \infty,$$

and generates a unique strong solution to (10). In the unconstrained benchmark, total wealth defined below remains positive. In the constrained model, $X_t \geq \underline{x}$ and $a_t \in \mathcal{K}(X_t, Y_t)$. The negative part of terminal utility is uniformly integrable.

The solvency condition must be stated in total rather than financial wealth in a complete market, because future earnings can support current borrowing. In an incomplete market, human capital cannot generally be pledged at its full arbitrage value; then a financial borrowing limit is economically necessary.

Before deriving the dynamic programme, it is useful to exploit the homogeneity of the state equations and CRRA preferences.

The diffusion system is homogeneous in (x,y,c,a). This gives a useful dimension reduction.

Proposition 2 (Homogeneity). *Suppose the constraint correspondence is conic:* $\mathcal{K}(kx,ky) = k\mathcal{K}(x,y)$ *for every* $k > 0$. *Then* $V(t,kx,ky) = k^{1-\gamma}V(t,x,y)$. *Consequently, for* $y > 0$ *there exists a function* v *such that*

$$V(t,x,y) = y^{1-\gamma}v(t,z), \qquad z = x/y.$$

Proof. If (c,a) is admissible from (x,y), then (kc,ka) is admissible from (kx,ky) and the associated state is multiplied by k. CRRA utility is homogeneous of degree $1-\gamma$. Taking suprema in both directions proves (15); setting $k = 1/y$ proves (16).

The wealth-to-income ratio z is therefore the only endogenous spatial state when coefficients are constant. This observation is central to the numerical method in Section 10.

Dynamic Programming and the HJB Equation

For a smooth test function $\varphi(t,x,y)$, the controlled generator is

$$\begin{aligned}\mathcal{L}^{c,a}\varphi = \;& \varphi_t + (rx + a^\top b + y - c)\varphi_x \\ & + \alpha_Y y\varphi_y + \tfrac{1}{2}a^\top C a\,\varphi_{xx} \\ & + y\,a^\top \Sigma\eta\,\varphi_{xy} + \tfrac{1}{2}y^2\|\eta\|^2\varphi_{yy}.\end{aligned}$$

The cross derivative is essential. It transmits the correlation between portfolio returns and real-income innovations into optimal demand.

Under the dynamic programming principle, the value function formally solves

$$\begin{aligned}0 = \;& V_t + \sup_{c>0,\, a\in\mathcal{K}(x,y)}\{U(c) - \delta V \\ & + (rx + a^\top b + y - c)V_x + \alpha_Y y V_y \\ & + \tfrac{1}{2}a^\top C a V_{xx} + y\,a^\top\Sigma\eta V_{xy} \\ & + \tfrac{1}{2}y^2\|\eta\|^2 V_{yy}\},\end{aligned}$$

with terminal condition

$$V(T,x,y) = \beta U(x).$$

If $V_x > 0$ and $V_{xx} < 0$, the unconstrained first-order conditions are

$$\begin{aligned} c^* &= V_x^{-1/\gamma}, \\ a^* &= -C^{-1}\left(b\frac{V_x}{V_{xx}} + y\Sigma\eta\frac{V_{xy}}{V_{xx}}\right).\end{aligned}$$

The first term in (21) is the conventional risk-premium demand. The second is the hedge against changes in the marginal value of wealth caused by income shocks. Even with constant expected returns, labour income creates a state-dependent hedging portfolio.

Proposition 3 (Concavity and monotonicity). *If the admissible set is convex and the state constraints are convex, then* $V(t,\cdot,\cdot)$ *is jointly concave and nondecreasing in financial wealth. If additional income can be saved without reducing feasibility,* V *is also nondecreasing in* y.

Proof. For two initial states and admissible controls, a convex combination of the controls generates the corresponding convex combination of state processes because (3)–(10) are linear in levels and controls. Concavity of U, followed by expectation and supremum, proves concavity. Monotonicity follows by investing any additional initial resource in the real account and following the original policy.

Smoothness need not hold at active portfolio or borrowing constraints. In that case (18) is interpreted in the viscosity sense, using the comparison framework of Crandall, Ishii and Lions [17] and the stochastic control treatment of Fleming and Soner [18].

Complete Market Closed-Form Solution

The complete-market analysis begins by assigning a risk-adjusted market value to the future labour-income stream.

Assume $n = m$, Σ is nonsingular and every component of the real-income shock is spanned by traded assets. Under the risk-neutral measure $\mathbb{Q}$, defined by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(-\lambda^\top W_t - \tfrac{1}{2}\|\lambda\|^2 t\right), \quad W_t^{\mathbb{Q}} = W_t + \lambda t,$$

real income has drift

$$\alpha_Y - \eta^\top\lambda$$

Its market value is

$$H(t,y) = \mathbb{E}_{t,y}^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)}\, Y_s\,\mathrm{d}s\right] = h(t)y,$$

where

$$\begin{aligned} \kappa &= r - \alpha_Y + \eta^\top \lambda, \\ h(t) &= \begin{cases} \frac{1-e^{-\kappa(T-t)}}{\kappa}, & \kappa \neq 0, \\ T - t, & \kappa = 0. \end{cases} \end{aligned}$$

The function h satisfies

$$h'(t) + 1 + (\alpha_Y - \eta^\top \lambda - r)h(t) = 0, \qquad h(T) = 0.$$

The term $\eta^\top \lambda$ is the risk adjustment to expected income growth. Income that covaries positively with priced risks has a lower market value than a naive discounted expectation. This is also the channel through which inflation exposure affects human wealth.

This human-capital value permits financial wealth and labour income to be combined into a single total-wealth state.

Define total wealth

$$Z_t = X_t + H(t, Y_t) = X_t + h(t)Y_t.$$

The risky-asset portfolio replicating human wealth is

$$a_t^H = (\Sigma^\top)^{-1} h(t) Y_t \eta,$$

because

$$(a_t^H)^\top \Sigma = h(t) Y_t \eta^\top$$

Applying Itô's formula and using (26) gives

$$\mathrm{d}Z_t = (rZ_t + \tilde{a}_t^\top b - c_t)\,\mathrm{d}t + \tilde{a}_t^\top \Sigma\,\mathrm{d}W_t, \qquad \tilde{a}_t = a_t + a_t^H.$$

Thus total wealth follows the standard Merton dynamics. Financial asset demand equals the optimal total-wealth portfolio minus the implicit risky exposure of human capital. The transformed problem admits an explicit consumption and portfolio policy.

Let

$$q = \frac{\delta - (1-\gamma)\left(r + \frac{\|\lambda\|^2}{2\gamma}\right)}{\gamma}.$$

Define the annuity factor

$$F(t) = \begin{cases} \frac{1}{q} + \left(\beta^{1/\gamma} - \frac{1}{q}\right) e^{-q(T-t)}, & q \neq 0, \\ \beta^{1/\gamma} + T - t, & q = 0. \end{cases}$$

It satisfies

$$F'(t) = qF(t) - 1$$

and

$$F(T) = \beta^{1/\gamma}$$

Theorem 4 (Closed-form complete-market policy). *Suppose Assumption 4.1 holds, $Z_t > 0$, C is positive definite and all income shocks are spanned. Then the value function and optimal controls are*

$$\begin{aligned} V(t,x,y) &= \frac{F(t)^\gamma}{1-\gamma}[x + h(t)y]^{1-\gamma}, \\ c_t^* &= \frac{Z_t}{F(t)}, \\ a_t^* &= \frac{Z_t}{\gamma} C^{-1} b - (\Sigma^\top)^{-1} h(t) Y_t \eta. \end{aligned}$$

The optimal real saving flow, and hence the income-normalised saving rate, is obtained by substituting (33)–(34) into (11).

Proof. Equation (29) reduces the problem to a complete-market consumption–portfolio problem in Z. Use the candidate $\hat{V}(t,z) = A(t)z^{1-\gamma}/(1-\gamma)$. The first-order conditions give $c = A^{-1/\gamma} z$ and $\tilde{a} = zC^{-1}b/\gamma$. Substitution into the HJB equation yields

$$\begin{aligned} A' &+ \left[(1-\gamma)\left(r + \frac{\|\lambda\|^2}{2\gamma}\right) - \delta\right] A \\ &+ \gamma A^{-(1-\gamma)/\gamma} = 0, \qquad A(T) = \beta. \end{aligned}$$

Setting $A = F^\gamma$ produces $F' = qF - 1$ and (31). Subtracting the replicating exposure (28) from the optimal total risky portfolio gives (34). The candidate is increasing and concave in z. Standard localisation, Itô's formula and the supermartingale verification argument show that it dominates the payoff of every admissible policy, with equality for the displayed controls.

To interpret the solution, the optimal financial portfolio is separated into speculative and human-capital-hedging components.

Equation (34) decomposes into

$$\underbrace{\frac{Z_t}{\gamma} C^{-1} b}_{\text{speculative demand on total wealth}} \qquad - \qquad \underbrace{(\Sigma^\top)^{-1} h(t) Y_t \eta.}_{\text{human-capital hedge}}$$

The first component is increasing in total wealth and the Sharpe-price vector, and decreasing in risk aversion. The second does not disappear as $\gamma \to \infty$ because it is a hedge against an endowed exposure rather than a speculative position. Near retirement, $h(t) \to 0$ and financial demand converges to the standard Merton rule based on financial wealth alone.

For a single risky asset with real excess return b, volatility σ and real-income volatility η, (34) becomes

$$\frac{a_t^*}{X_t} = \frac{b}{\gamma\sigma^2}\left(1 + \frac{h(t)Y_t}{X_t}\right) - \frac{h(t)Y_t}{X_t}\frac{\eta}{\sigma}.$$

The first term is the leveraged total-wealth demand. The second offsets the income exposure. A young household with bond-like labour income may optimally hold a high equity share, whereas a household whose income is equity-like should hold less equity. This distinction cannot be recovered from age alone.

Corollary 5 (Consumption response to human wealth). *At fixed (t,x), optimal consumption increases linearly with income:* $\frac{\partial c^*}{\partial y} = \frac{h(t)}{F(t)} > 0$. *Moreover, a shock that reduces risk-adjusted human wealth by one unit reduces current consumption by* $1/F(t)$ *units.*

Corollary 6 (Retirement limit). *As* $t \uparrow T$, $h(t) \downarrow 0$, $Z_t \to X_t$, *and the human-capital hedge vanishes. If* $\beta > 0$,

$$c_T^- = X_T/\beta^{1/\gamma}.$$

Incomplete Markets and Unspanned Income Risk

Complete markets are analytically useful but empirically demanding. A large part of earnings risk is occupation-specific, health-related or otherwise unspanned. To represent this fact, decompose the real-income volatility into traded and orthogonal components. Let

$$W = (W^M, W^\perp),$$

where W^M drives traded returns and $W^\perp$ is independent. Write

$$\frac{\mathrm{d}Y_t}{Y_t} = \alpha_Y\,\mathrm{d}t + \eta_M^\top\,\mathrm{d}W_t^M + \sigma_\perp\,\mathrm{d}W_t^\perp, \qquad \sigma_\perp \geq 0.$$

Only η_M is hedgeable. When $\sigma_\perp > 0$, future income has no unique arbitrage price and cannot simply be added to financial wealth. The household's subjective valuation depends on preferences, liquidity and constraints.

Using Proposition 4.2, set $z = x/y$, normalised consumption $u = c/y$ and normalised risky positions $p = a/y$. Straight differentiation of

$$V = y^{1-\gamma} v(t,z)$$

gives

$$\begin{aligned} V_x &= y^{-\gamma} v_z, & V_{xx} &= y^{-1-\gamma} v_{zz}, \\ V_y &= y^{-\gamma}[(1-\gamma)v - z v_z], & V_{xy} &= y^{-1-\gamma}[-\gamma v_z - z v_{zz}], \end{aligned}$$

and

$$V_{yy} = y^{-1-\gamma}\{-\gamma(1-\gamma)v + 2\gamma z v_z + z^2 v_{zz}\}.$$

Substitution into (18) produces the reduced HJB equation

$$\begin{aligned} 0 = \quad & v_t - \delta v + \alpha_Y[(1-\gamma)v - z v_z] \\ & + \tfrac{1}{2}\left(\|\eta_M\|^2 + \sigma_\perp^2\right)[-\gamma(1-\gamma)v \\ & \qquad + 2\gamma z v_z + z^2 v_{zz}] \\ & + \sup_{u>0,\, p\in\mathcal{K}_z} \{\tfrac{u^{1-\gamma}}{1-\gamma} + (rz + p^\top b + 1 - u)v_z \\ & \qquad + \tfrac{1}{2} p^\top C p\, v_{zz} + p^\top \Sigma \eta_M \\ & \qquad \times (-\gamma v_z - z v_{zz})\}. \end{aligned}$$

Its terminal condition is

$$v(T,z) = \frac{\beta z^{1-\gamma}}{1-\gamma}.$$

The normalised controls are

$$\begin{aligned} u^* &= v_z^{-1/\gamma}, \\ p^* &= -C^{-1}\left[b\,\frac{v_z}{v_{zz}} + \Sigma\eta_M \frac{-\gamma v_z - z v_{zz}}{v_{zz}}\right] \end{aligned}$$

when portfolio constraints do not bind.

The unspanned volatility $\sigma_\perp$ enters through the curvature of the value function. It has no direct hedge portfolio, but it raises the precautionary value of liquid wealth for prudent preferences. CRRA utility has

$$U'''(c) = \gamma(\gamma+1)c^{-\gamma-2} > 0,$$

so a mean-preserving increase in uninsurable income risk generally depresses current consumption and increases buffer saving. This mechanism is consistent with the precautionary-saving models of Deaton [19] and Carroll [20].

Proposition 7 (Complete-market consistency). *If* $\sigma_\perp = 0$, Σ *is square and nonsingular, and there are no constraints, then the reduced solution implied by (32) satisfies (39):*

$$v(t,z) = \frac{F(t)^\gamma}{1-\gamma}[z + h(t)]^{1-\gamma}.$$

Proof. Substitute $x = zy$ into (32) and factor out $y^{1-\gamma}$. The derivatives of (43), together with the ODEs for F and h, cancel every coefficient in (39). The maximisers reduce to (33) and (34) after multiplication by y. $\square$

Proposition 7.1 is also a computational unit test: a solver for the incomplete model should converge to (43) as $\sigma_\perp \downarrow 0$ and the constraints are relaxed.

Borrowing and Portfolio Constraints

Households cannot ordinarily borrow the full market value of future earnings, and many face short-sale or leverage restrictions. The dynamics of the wealth-to-income ratio are needed to formulate a mathematically viable lower boundary. Applying Itô's formula to $z_t = X_t/Y_t$, with $u_t = c_t/Y_t$ and $p_t = a_t/Y_t$, gives

$$\begin{aligned} \mathrm{d}z_t = \quad & \mu_z(z_t, u_t, p_t)\,\mathrm{d}t \\ & + (p_t^\top \Sigma - z_t \eta_M^\top)\,\mathrm{d}W_t^M \\ & - z_t \sigma_\perp\,\mathrm{d}W_t^\perp, \\ \mu_z(z,u,p) = \quad & 1 + \left(r - \alpha_Y + \|\eta_M\|^2 + \sigma_\perp^2\right)z \\ & + p^\top b - u - p^\top \Sigma \eta_M. \end{aligned}$$

Let the feasible normalised portfolio set be

$$\mathcal{K}_z = \{p \in \mathbb{R}^n : \ell_i(z) \leq p_i \leq u_i(z), \quad \mathbf{1}^\top p \leq z + \bar{d}\},$$

where $\bar{d} \geq 0$ is the maximum debt-to-income ratio. A no-short-sale rule sets $\ell_i = 0$; an exposure cap may set

$$u_i(z) = \omega_i z^+.$$

When $\bar{d} > 0$, define the default time

$$\tau_D = \inf\{s \geq t : z_s \leq -\bar{d}\} \wedge T.$$

The control problem is stopped at τ_D and assigned a specified recovery value $v_D(t)$ at $z = -\bar{d}$. This absorbing-boundary formulation is required because a negative hard state constraint is not viable when $\sigma_\perp > 0$. For fixed derivatives (v_z, v_{zz}) with $v_{zz} < 0$, the portfolio step in (39) is a strictly concave quadratic programme:

$$\max_{p\in\mathcal{K}_z} \left\{\tfrac{1}{2} v_{zz} p^\top C p + p^\top[b v_z + \Sigma\eta_M(-\gamma v_z - z v_{zz})]\right\}.$$

Thus constraints do not require a search over arbitrary strategies; each policy update solves a small convex-optimisation problem after sign reversal. The Karush–Kuhn–Tucker conditions are

$$\begin{aligned} v_{zz}Cp + bv_z - A^{\top}\xi &= -\Sigma\eta_M(-\gamma v_z - zv_{zz}), \\ Ap - d(z) &\leq 0, \qquad \xi \geq 0, \\ \xi^{\top}[Ap - d(z)] &= 0. \end{aligned}$$

Indeed, viability of a hard boundary $z = z_{\min}$ requires both the diffusion to be tangent and the drift to point inward:

$$p^{\top}\Sigma - z_{\min}\eta_M^{\top} = 0, \qquad z_{\min}\sigma_{\perp} = 0, \\ \mu_z(z_{\min}, u, p) \geq 0.$$

For $\sigma_{\perp} > 0$, the second equality in (51) forces $z_{\min} = 0$. Hence the no-unsecured-borrowing case is the natural state-constraint specification; at $z = 0$, one may take $p = 0$ and require $u \leq 1$. For $\bar{d} > 0$, the absorbing default boundary (47) is used instead of an artificial reflecting boundary, which would inject resources into the household.
Remark 8 (Constraint-induced nonlinearity). Even with CRRA preferences and constant investment opportunities, the risky share need not be constant. A borrowing limit binds most strongly when financial wealth is low relative to income, while a no-short-sale rule can prevent the household from offsetting equity like human capital. The resulting policy is piecewise smooth, with kinks at KKT switching surfaces.

Comparative Statics
We first examine the preference parameters governing aversion to risk and the timing of consumption.
Holding total wealth fixed, the speculative component in (35) satisfies

$$\frac{\partial}{\partial\gamma}\left(\frac{Z}{\gamma}C^{-1}b\right) = -\frac{Z}{\gamma^2}C^{-1}b.$$

Greater risk aversion reduces speculative exposure in every direction with a positive mean variance demand. It does not mechanically scale the replicating human-capital hedge, although it changes consumption and the future state distribution.
Impatience affects the consumption annuity through q. Since

$$\frac{\partial q}{\partial\delta} = \frac{1}{\gamma} > 0,$$

a larger discount rate generally raises the current consumption-to-total-wealth ratio. For

$$\tau = T - t,$$

$$\begin{aligned} \frac{\partial c^*}{\partial\delta} &= -\frac{Z}{\gamma F^2}\frac{\partial F}{\partial q}, \\ \frac{\partial F}{\partial q} &= -\frac{1-e^{-q\tau}}{q^2} - \tau\left(\beta^{1/\gamma} - \frac{1}{q}\right)e^{-q\tau}. \end{aligned}$$

When the bequest motive is not excessively large, current consumption rises with impatience.
We next isolate the channels through which inflation changes real income, expected real returns and the household's hedging demand.
Inflation influences four objects simultaneously:

$$\alpha_Y, \qquad \eta = \sigma_L - \sigma_P, \qquad b, \qquad \Sigma.$$

Therefore, changing expected inflation while holding every real return and real income parameter fixed studies only a relabelling of nominal units. A genuine inflation risk experiment must change σ_P or the covariance structure in (6). From (25), for $\tau > 0$,

$$\frac{\partial h}{\partial\kappa} = \frac{e^{-\kappa\tau}(1+\kappa\tau)-1}{\kappa^2} \leq 0,$$

with limit

$$\frac{-\tau^2}{2}$$

at

$$\kappa = 0$$

Hence an increase in the priced-risk adjustment $\eta^{\top}\lambda$ lowers human wealth and current consumption. If nominal wages are imperfectly indexed, an increase in inflation volatility typically raises $\|\eta\|$ and can reduce financial risk bearing capacity.
The remaining earnings horizon provides a second source of variation in human wealth and portfolio demand.
Differentiating (25) with respect to remaining working life gives

$$\frac{\partial h}{\partial\tau} = e^{-\kappa\tau} > 0.$$

A longer earnings horizon increases total wealth and the speculative demand based on total resources. It also increases the human-capital hedge. The net effect on a particular asset is

$$\frac{\partial a^*}{\partial h} = \frac{Y}{\gamma}C^{-1}b - (\Sigma^{\top})^{-1}Y\eta.$$

Thus the familiar prediction that young investors should hold more equity is conditional: it holds when labour income is sufficiently bond-like, but can be reversed when earnings are highly exposed to equity shocks.
Finally, the asset income correlation determines the hedgeable component, whereas the orthogonal component generates precautionary saving.
In the one-asset model, holding the expected real excess return fixed,

$$\frac{\partial a^*}{\partial\eta} = -\frac{hY}{\sigma}.$$

A more positive earnings asset correlation reduces the asset holding. Under a no short sale constraint, the response stops at zero and leaves residual income risk unhedged.
There is generally no closed-form derivative with respect to $\sigma_{\perp}$. Under standard prudence conditions, greater independent unspanned risk raises the marginal value of liquid wealth and reduces normalised consumption. Numerically, test

$$\Delta_u(z,t) = u^*(z,t;\sigma_{\perp,2}) - u^*(z,t;\sigma_{\perp,1}), \qquad \sigma_{\perp,2} > \sigma_{\perp,1}.$$

The expected sign is

$$\Delta_u < 0$$

away from boundaries where constraints force consumption.

Numerical Method and Reproducible Design

The numerical construction begins with a bounded approximation to the reduced wealth to income state space. The reduced state variable

$$z = \frac{x}{y}$$

is defined on

$$[z_{\min}, z_{\max}],$$

where $z_{\min} = 0$ for the viable no-borrowing state constraint. In experiments that permit unsecured debt, $z_{\min} = -\bar{d}$ is an absorbing default boundary with

$$v(t, -\bar{d}) = v_D(t).$$

The upper endpoint $z_{\max}$ is chosen sufficiently large. Time is discretised backward on

$$t_k = k\Delta t,$$

and the spatial grid is

$$z_j = z_{\min} + j\Delta z$$

At large z, labour income is negligible relative to financial wealth, motivating the asymptotic boundary

$$v(t, z) \sim \frac{F(t)^\gamma}{1-\gamma} z^{1-\gamma}, \qquad z \to \infty.$$

This condition is implemented as a one-sided derivative or by matching the Merton asymptote. At $z_{\min} = 0$, controls satisfy the viability conditions (51); at $z_{\min} = -\bar{d}$, the absorbing condition (60) is imposed.

On this domain, the nonlinear HJB equation is solved by an implicit monotone policy iteration procedure.

At each backward time step, initialise a feasible policy $\left(u_j^{(0)}, p_j^{(0)}\right)$. Conditional on policy iteration k, freeze the controls and solve

$$\frac{v_j^n - v_j^{n+1}}{\Delta t} + \mathcal{D}_h^{u^{(k)}, p^{(k)}} v_j^n + \frac{\left(u_j^{(k)}\right)^{1-\gamma}}{1-\gamma} = 0,$$

where $\mathcal{D}_h$ discretises the controlled spatial operator. Diffusion is approximated by centred second differences. The first derivative uses upwinding according to the sign of the controlled drift. This makes the discrete operator an M-matrix under standard step-size and coefficient conditions.

Update consumption analytically using (41). Update the portfolio by solving (48) with the current finite differences. Iterate until

$$\max_j \left(\left| u_j^{(k+1)} - u_j^{(k)} \right| + \left\| p_j^{(k+1)} - p_j^{(k)} \right\| \right) < \varepsilon_{\mathrm{PI}}.$$

Howard policy iteration is attractive because the control update respects constraints exactly and usually converges faster than value iteration.

Monotonicity, stability and consistency are the key ingredients for convergence to the viscosity solution. The general approximation theory for controlled diffusions is developed by Kushner and Dupuis [21]. High order centred schemes may look more accurate on smooth test cases but can violate monotonicity near constraint induced kinks.

Table 1 specifies a transparent benchmark in annual real units. It is not an estimate for a named country or household. The two risky assets may be interpreted as real equity and a nominal bond portfolio expressed in real terms. Their volatility matrix already incorporates price index risk.

Table 1: Illustrative benchmark parameters.

Parameter	Value	Interpretation
T	30	remaining planning horizon in years
r	0.020	real short rate
δ	0.030	subjective discount rate
γ	5	relative risk aversion
β	1	terminal bequest weight
b	(0.040, 0.010)$^{\mathsf{T}}$	real excess-return vector
Σ	[[0.18, −0.0^{2}], [0.04, 0.0^{8}]]	real-return volatility matrix
αY	0.015	expected real-income growth
ηM	(0.08, 0.04)$^{\mathsf{T}}$	spanned real-income exposure
σ⊥	0.10	unspanned income volatility
d̄	0	no-unsecured-borrowing benchmark
zmax	40	upper numerical wealth–income boundary

For this matrix,

$$\lambda = \Sigma^{-1} b = (0.223684, 0.013158)^{\mathsf{T}}$$

and

$$\|\lambda\|^2 = 0.050208$$

The complete market benchmark therefore has

$$q = 0.026017, \quad \kappa = r - \alpha_Y + \eta_M^{\mathsf{T}} \lambda = 0.023421.$$

These values make the analytic calculations independently checkable.

The benchmark is extended through a sequence of nested computational scenarios that isolate market incompleteness, constraints, inflation and preference effects.

The computational study should compare the following nested cases:

1. complete markets, no constraints and $\sigma_\perp = 0$;
2. unspanned income risk with $\sigma_\perp \in \{0.05, 0.10, 0.20\}$;
3. no borrowing ($\bar{d} = 0$) and absorbing default thresholds

$\bar{d} \in \{0.5,1\}$ with a stated recovery value v_D;
4. no short sales and asset caps $0 \le p_i \le 2z^+$;
5. low, benchmark and high inflation exposure obtained by scaling σ_P while recomputing $(\alpha_Y, \eta, b, \Sigma)$ from nominal primitives;
6. retirement horizons $T - t \in \{5,10,20,30\}$ and risk aversion $\gamma \in \{2,5,10\}$.

The primary outputs are the consumption to income ratio u^*, saving rate, risky positions p_i^*, constraint multipliers and certainty-equivalent wealth. Welfare differences are the percentage increase in initial financial wealth required to make the household indifferent between policies.

Analytic Benchmark Results and Interpretation

We begin the interpretation of the benchmark with the annuity and human-capital factors implied by the closed form solution.

Using (31) and (25), Table 2 gives exact complete market factors for the illustrative parameters. These are evaluations of closed form formulas, not estimates or Monte Carlo results.

Table 2: Closed form factors by remaining horizon.

T − t	h(t)	F(t)	c*/Z = 1/F(t)
5	4.7183	5.5665	0.1796
10	8.9153	9.5759	0.1044
20	15.9690	16.1873	0.0618
30	21.5498	21.2842	0.0470

As the horizon lengthens, human wealth rises and the household spreads total resources over more consumption dates, lowering the instantaneous consumption to total wealth ratio. Since financial wealth may be small relative to hY, the consumption to financial wealth ratio can nevertheless be high for a young household. This is not imprudence in the complete market: future earnings are fully pledgeable there.

The portfolio decomposition can then be evaluated at a representative wealth income state.

Take annual real income as numeraire, $Y = 1$, and financial wealth $X = 5$ at a 30 year horizon. Then

$$Z = 26.5498$$

The speculative total wealth position is

$$\frac{Z}{\gamma} C^{-1} b = \frac{Z}{\gamma} (\Sigma^\top)^{-1} \lambda,$$

while the hedge is

$$(\Sigma^\top)^{-1} h\eta$$

This calculation illustrates why reporting only portfolio shares of financial wealth can be misleading: the denominator omits the household's largest asset, human capital. A high gross position need not be purely speculative; part may offset the risk in future earnings.

The consumption rule gives

$$c^* = Z/F = 1.2474$$

real income units per year. This amount exceeds current labour income because the unconstrained complete-market household can annuitise human wealth. Under unspanned income risk or a debt limit, this policy is generally infeasible, and the numerical solution lowers consumption at low z.

The same decomposition clarifies the contribution of inflation hedging assets. Suppose an asset's nominal volatility becomes more positively aligned with σ_P. Its real volatility row $\sigma_i^N - \sigma_P$ shrinks, potentially making the asset a better purchasing power hedge. At the same time, its real expected return changes by the covariance correction in (6). The optimal response is therefore not determined by volatility alone. A suitable inflation-linked asset can raise welfare by reducing portfolio variance, spanning more of η, and increasing the collateral value of human capital.

The economic importance of these mechanisms is summarised by the welfare loss generated by restricted portfolio rules.

Let

$$V^*(t, x, y)$$

denote optimal welfare and

$$V^R(t, x, y)$$

welfare under a restricted rule, such as a fixed 60–40 portfolio. Define equivalent financial wealth variation ω by

$$V^*(t, x, y) = V^R(t, (1 + \omega)x, y).$$

In the homogeneous model, ω is found by one dimensional root finding. It should be reported by wealth to income ratio. A single average welfare cost conceals that fixed share rules are most distorted when human capital is large and correlated with traded risk.

Model Verification, Sensitivity and Robustness

Verification begins by comparing the numerical solution with the analytic complete market benchmark. A credible implementation should pass five tests. First, with

$$\sigma_\perp = 0$$

and constraints relaxed, numerical v, u and p must converge to (43), (33) and (34). Second, the discrete HJB residual

$$\mathcal{R}_h = \max_{n,j} \left| \mathcal{H}_h(v_j^n, u_j^n, p_j^n) \right|$$

must decrease with grid refinement. Third,

$$v_z > 0$$

and

$$v_{zz} < 0$$

should hold away from boundaries. Fourth, simulated controls must respect every constraint. Fifth, welfare under the computed optimum must weakly dominate welfare under feasible perturbations. Use at least three grids, for example

$$(N_t, N_z) = (300{,}400), (600{,}800), (1200{,}1600)$$

For a quantity G_h, the successive grid difference

$$E_h = \|G_h - G_{h/2}\|_{\infty,\mathcal{D}_0}$$

is evaluated on an interior domain $\mathcal{D}_0$ to distinguish discretisation error from truncation error. Stability with respect to $z_{\max}$ must also be reported. The converged feedback policies are subsequently checked by forward Monte Carlo simulation.

After solving the HJB equation, interpolate the controls and simulate (3) and (10) with correlated increments. Estimate

$$\hat{J}_M = \frac{1}{M}\sum_{k=1}^{M}\left[\sum_{n=0}^{N_t-1} e^{-\delta t_n} U\left(c_{n,k}\right)\Delta t + e^{-\delta T}\beta U\left(X_{T,k}\right)\right].$$

Antithetic variates and common random numbers reduce noise in policy comparisons. A 95% Monte Carlo interval is

$$\hat{J}_M \pm 1.96\hat{s}_J/\sqrt{M}$$

Paths that violate solvency because of time discretisation must not be discarded silently; the step size or boundary treatment should be corrected.

Sensitivity analysis and empirical identification complete the validation framework.

Local sensitivities can be approximated by symmetric differences,

$$S_{G,\theta} = \frac{\theta}{G(\theta)}\frac{G(\theta+\varepsilon\theta)-G(\theta-\varepsilon\theta)}{2\varepsilon\theta},$$

for

$$\varepsilon \in [10^{-3}, 10^{-2}]$$

Global sensitivity should vary γ, δ, income growth, inflation exposure, return premia and correlations jointly. Correlation matrices must remain positive semidefinite; varying pairwise correlations independently can create an invalid diffusion.

An empirical application requires nominal wage histories, consumption or liquid saving, asset returns and a price index. Equations (1)–(2) should be estimated jointly or with covariance restrictions preserved. If observations are annual, exact Gaussian transition densities for log P and log L are preferable to Euler regressions. The preference parameters (γ, δ) are not identified from returns alone; consumption data, portfolio shares or external restrictions are needed.

Measurement error in income is consequential because reporting noise can be mistaken for unspanned economic risk. A state space specification can separate persistent real income shocks from observation error. Parameter uncertainty may then be propagated by solving the control problem across posterior draws instead of plugging in a single estimate.

Actuarial and Policy Implications

The model has direct implications for retirement planning, pension design and actuarial asset liability management. A defined contribution balance represents only the funded component of retirement resources. The household's broader economic balance sheet also includes the risk adjusted value of future labour income and any inflation linked pension entitlements. Let X_t denote real financial wealth, Y_t real labour income and

$$H(t, Y_t) = h(t)Y_t$$

the market consistent value of future earnings. Total real wealth is therefore

$$Z_t = X_t + H(t, Y_t) = X_t + h(t)Y_t,$$

where

$$\begin{aligned} h(t) &= \begin{cases} \frac{1-\exp[-\kappa(T-t)]}{\kappa}, & \kappa \neq 0, \\ T-t, & \kappa = 0, \end{cases} \\ \kappa &= r - \alpha_Y + \eta^\top\lambda. \end{aligned}$$

Here, r is the real interest rate, α_Y is expected real income growth, η is the vector of labour income risk exposures and λ is the market price of risk vector. Since

$$\frac{\partial h(t)}{\partial\kappa} = \frac{e^{-\kappa(T-t)}[1+\kappa(T-t)]-1}{\kappa^2} \leq 0,$$

human capital decreases when labour income is more strongly exposed to priced systematic risk. Under complete markets, the optimal real investment in risky assets is

$$a_t^* = \frac{Z_t}{\gamma}C^{-1}b - (\Sigma^\top)^{-1}h(t)Y_t\eta,$$

where γ is relative risk aversion, b is the real excess return vector, Σ is the volatility matrix and

$$C = \Sigma\Sigma^\top$$

The first term in (74) is the speculative demand based on total wealth, while the second is the hedge against human capital risk. Dividing by financial wealth gives

$$\frac{a_t^*}{X_t} = \frac{1+\chi_t}{\gamma}C^{-1}b - \chi_t(\Sigma^\top)^{-1}\eta, \qquad \chi_t = \frac{h(t)Y_t}{X_t}.$$

Consequently, age and financial wealth are not sufficient statistics for portfolio choice. A worker whose income is highly correlated with equity returns already possesses equity like human capital and should hold less financial equity than a worker with stable, bond-like earnings. In particular, if

$$\mathrm{Cov}_t\left(\frac{\mathrm{d}Y_t}{Y_t}, \frac{\mathrm{d}S_t}{S_t}\right) = \eta^\top\sigma_S \, \mathrm{d}t > 0,$$

then labour income and equity wealth tend to decline simultaneously, increasing the required hedging adjustment. This explains why a worker in a cyclical industry may optimally hold less equity than a public sector worker of the same age, income and accumulated pension wealth.

Inflation protection must be considered jointly with wage indexation. Suppose the price index P_t and nominal labour income L_t satisfy

$$\frac{dP_t}{P_t} = \pi_P \ dt + \sigma_P^\top dW_t, \qquad \frac{dL_t}{L_t} = \mu_L \ dt + \sigma_L^\top dW_t.$$

Since

$$Y_t = \frac{L_t}{P_t},$$

real-income exposure is

$$\eta = \sigma_L - \sigma_P.$$

Incomplete wage indexation therefore increases the household's exposure to unexpected inflation. For a nominal asset with volatility σ_i^N, its real volatility is

$$\sigma_i^R = \sigma_i^N - \sigma_P.$$

An inflation linked security is valuable because it reduces the covariance between real retirement assets and inflation sensitive consumption liabilities.
Let L_t^P denote the actuarial value of future real pension liabilities and suppose its diffusion has real-currency exposure $\ell_t^P \in \mathbb{R}^m$,

$$dL_t^P = \mu_L^P(t)\,dt + (\ell_t^P)^\top\,dW_t.$$

Define the household or pension-fund surplus by

$$\mathcal{S}_t = X_t + h(t)Y_t - L_t^P.$$

The instantaneous surplus variance is therefore

$$\frac{1}{dt}\mathrm{Var}_t(\,d\mathcal{S}_t) = \|\Sigma^\top a_t + h(t)Y_t\eta - \ell_t^P\|^2.$$

When all relevant risks are spanned, the minimum variance liability-hedging portfolio is

$$a_t^{\mathrm{LHP}} = (\Sigma^\top)^{-1}[\ell_t^P - h(t)Y_t\eta].$$

If indexed securities are unavailable, the unhedgeable basis risk component is

$$\varepsilon_t^\perp = \left[I - \Pi_{\mathrm{Im}(\Sigma^\top)}\right][\ell_t^P - h(t)Y_t\eta],$$

where $\Pi_{\mathrm{Im}(\Sigma^\top)}$ is the orthogonal projection onto the space of traded risks. The quantity $\| \varepsilon_t^\perp \|^2$ measures the residual inflation and wage basis risk faced by the household or pension fund.
The borrowing results distinguish illiquidity from low lifetime resources. A young household may satisfy $X_t \ll h(t)Y_t$ while still having substantial total wealth. Nevertheless, unspanned income risk prevents the complete-market value $h(t)Y_t$ from being treated as fully pledgeable collateral. If

$$\frac{dY_t}{Y_t} = \alpha_Y \ dt + \eta_M^\top\,dW_t^M + \sigma_\perp\,dW_t^\perp,$$

then a stress adjusted borrowing limit may be specified as

$$\begin{aligned} D_t &\le \vartheta H_t^{\text{pledge}}, \\ H_t^{\text{pledge}} &= h(t)Y_t \exp\left[-q_\alpha \sigma_\perp \sqrt{T-t}\right], \\ 0 &\le \vartheta < 1, \end{aligned}$$

where $D_t = (-X_t)^+$, q_α is a lower-tail risk quantile and ϑ is the fraction of stress adjusted human capital that may be pledged. The limit tightens as uninsurable income volatility increases. For insurers and pension administrators, the wealth-to-income ratio

$$z_t = \frac{X_t}{Y_t}$$

provides a useful segmentation variable, but it is not sufficient on its own. A more informative actuarial state vector is

$$\Xi_t = (z_t,\, \chi_t,\, \eta,\, \sigma_\perp,\, T-t,\, \gamma).$$

Default investment, consumption and borrowing policies can then be represented as a state-dependent mapping

$$\mathcal{G}(\Xi_t) = (c_t^*, a_t^*, D_t^{\max}).$$

The welfare effect of competing policies may be measured using certainty-equivalent wealth. If CE_t^P is the certainty equivalent under policy P, the proportional welfare gain of policy P_1 relative to P_0 is

$$\mathcal{W}_t(P_1, P_0) = \frac{\mathrm{CE}_t^{P_1}}{\mathrm{CE}_t^{P_0}} - 1.$$

The principal implication is that retirement and pension decisions should be based on the household's complete real balance sheet rather than on accumulated financial wealth alone. Human capital exposure, wage indexation, inflation risk, unspanned income volatility and pension liabilities jointly determine risk-bearing capacity. The framework therefore provides a quantitative basis for designing default pension portfolios, inflation hedges and prudent borrowing limits, while remaining a decision support model rather than a substitute for taxation, suitability regulation or individual financial advice.

Extensions

One natural extension allows inflation, interest rates and expected returns to depend on observable stochastic factors. Let K_t be observable factors following

$$dK_t = \mu_K(K_t)\,dt + \Gamma_K(K_t)\,dW_t.$$

Allow r, b, Σ, α_Y and η to depend on K_t. The value function becomes $V(t, x, y, k)$, and the HJB obtains derivatives in k and cross derivatives with wealth and income. These generate intertemporal hedging demands absent from the constant coefficient model. Affine factor specifications can preserve partial tractability, while homogeneity still removes one scale dimension.
A complementary extension represents abrupt changes in inflation and labour market conditions through a finite state

regime process. Inflation and labour markets may alternate between persistent regimes. Let M_t be a finite state Markov chain with generator

$$Q = (q_{ij})$$

The regime-specific value functions satisfy a coupled HJB system containing

$$\sum_{j\neq i} q_{ij}\left[V_j(t,x,y) - V_i(t,x,y)\right].$$

This can represent low-inflation, high inflation and recession states, but identification requires sufficient transitions and should not be attempted with a short sample merely because it improves in-sample fit. Mortality and annuitisation can be incorporated by introducing a survival intensity and death contingent utility.

With mortality intensity $\mu_d(t)$, survival weighted utility changes the effective discount rate to $\delta + \mu_d(t)$, while bequest utility enters at death. An actuarially fair annuity adds mortality credits but sacrifices liquidity and bequest. The control set may then include gradual annuitisation. This extension connects the model with stochastic life cycle pension analysis such as Cairns, Blake and Dowd [16].

Finally, transaction costs and discrete trading alter the continuous rebalancing assumption and lead to impulse or singular control formulations. Proportional transaction costs turn risky positions into additional state variables and create a no trade region. Davis and Norman [22] establish this structure for the classic consumption problem. With labour income and inflation, the centre of the no trade region moves with Y, the price index and the horizon. Discrete contributions and trading dates can instead be treated using impulse control or dynamic programming recursions.

Limitations

The model makes deliberate simplifications. Constant coefficients exclude predictable returns and time varying inflation risk. CRRA utility imposes homothetic responses and does not capture subsistence consumption, habit formation or ambiguity aversion. Labour income is lognormal and cannot become zero; unemployment would require jumps, a regime process or a separate employment state. Retirement is exogenous, taxes and housing are omitted, and the real locally riskless account may not be directly accessible to all households.

The complete-market formulas should not be used mechanically where earnings are non pledgeable. They are benchmarks identifying the roles of human capital and spanning. The incomplete-market HJB is more realistic but depends on boundary conditions, numerical resolution and calibrated constraints. Finally, an illustrative parameterisation demonstrates mechanisms only. An empirical application requires documented data, estimation uncertainty, out of sample validation and population specific institutions.

Conclusion

This paper developed a continuous-time consumption investment model in which household decisions respond jointly to inflation, stochastic labour income and uncertain asset returns. Starting from nominal price and wage dynamics,

$$\frac{dP_t}{P_t} = \pi_P\, dt + \sigma_P^\top\, dW_t, \qquad \frac{dL_t}{L_t} = \mu_L\, dt + \sigma_L^\top\, dW_t,$$

Itô's formula gives the real income process $Y_t = L_t/P_t$ as

$$\begin{aligned}\frac{dY_t}{Y_t} &= \alpha_Y\, dt + \eta^\top\, dW_t,\\ \alpha_Y &= \mu_L - \pi_P + \|\sigma_P\|^2 - \sigma_L^\top\sigma_P,\\ \eta &= \sigma_L - \sigma_P.\end{aligned}$$

Similarly, the real return of asset i has drift and volatility

$$\mu_i^R = \mu_i^N - \pi_P + \|\sigma_P\|^2 - (\sigma_i^N)^\top\sigma_P, \quad \sigma_i^R = \sigma_i^N - \sigma_P$$

Hence purchasing power risk depends not only on expected inflation, but also on inflation variance and its covariance with wages and asset payoffs.

The household selects real consumption c_t and risky positions a_t subject to

$$dX_t = (rX_t + a_t^\top b + Y_t - c_t)\, dt + a_t^\top \Sigma\, dW_t.$$

In a complete market, the risk adjusted value of future labour income is

$$\begin{aligned}H(t,Y_t) &= h(t)Y_t,\\ h(t) &= \frac{1-e^{-\kappa(T-t)}}{\kappa},\\ \kappa &= r - \alpha_Y + \eta^\top\lambda,\end{aligned}$$

with

$$h(t) = T - t$$

when

$$\kappa = 0$$

and

$$\lambda = \Sigma^{-1}b$$

Defining total wealth by

$$Z_t = X_t + h(t)Y_t,$$

the optimal controls are

$$\begin{aligned}c_t^* &= \frac{Z_t}{F(t)},\\ a_t^* &= \frac{Z_t}{\gamma}C^{-1}b - (\Sigma^\top)^{-1}h(t)Y_t\eta,\\ C &= \Sigma\Sigma^\top.\end{aligned}$$

The portfolio rule separates speculative demand based on total wealth from the replicating hedge required to offset human capital risk. When labour income contains an unspanned component,

$$\frac{dY_t}{Y_t} = \alpha_Y\, dt + \eta_M^\top\, dW_t^M + \sigma_\perp\, dW_t^\perp,$$

future earnings cannot be fully capitalised. The orthogonal risk $\sigma_\perp\, dW_t^\perp$ increases precautionary saving and makes the optimal policies state dependent. CRRA homogeneity reduces the problem to the wealth to income ratio,

$$V(t,x,y) = y^{1-\gamma}v(t,z), \qquad z = \frac{x}{y},$$

with normalised optimal consumption

$$\frac{c_t^*}{Y_t} = v_z(t,z)^{-1/\gamma}.$$

Borrowing, leverage and short sale restrictions are incorporated through a state dependent admissible set for the normalised portfolio. The principal implication is that age and accumulated financial wealth are not sufficient statistics for optimal household allocation. Risk bearing capacity depends on

$$\Xi_t = \left(\frac{X_t}{Y_t}, \frac{h(t)Y_t}{X_t}, \eta_M, \sigma_\perp, T-t, \gamma\right),$$

which combines liquidity, human capital, systematic income exposure, uninsurable risk, horizon and risk aversion. The model consequently provides a mathematically consistent benchmark for household portfolio choice, inflation hedging, pension design and actuarial asset liability management.

Declarations

Funding

This research received no specific grant from any funding agency in the public, commercial or not for profit sectors.

Conflict of Interest

The author declares that there are no financial or non-financial conflicts of interest that could have influenced the research reported in this article.

Data Availability

This study is theoretical and employs an illustrative numerical calibration; therefore, no observed or confidential household level data were collected or analysed. All parameter values and numerical inputs required to reproduce the analytical benchmark are provided within the article.

Ethics Approval

Ethical approval was not required because the study is based exclusively on mathematical analysis and illustrative numerical calculations and does not involve human participants, identifiable personal information or animals.

Consent to Participate

Not applicable.

Consent for Publication

Not applicable.

Author Contributions

P. N. Nangolo was responsible for the conceptualisation, methodology, mathematical formulation, formal analysis, numerical validation, interpretation of results, preparation of the original draft, and critical revision of the manuscript. The author read and approved the final manuscript.

Proof Details and Auxiliary Derivations

For completeness, the first auxiliary calculation derives the real-income process directly from the nominal primitives.
Let $f(l,p) = l/p$. Its derivatives are $f_l = p^{-1}$, $f_p = -lp^{-2}$, $f_{pp} = 2lp^{-3}$ and $f_{lp} = -p^{-2}$. Itô's formula gives

$$\begin{aligned} \mathrm{d}(L/P) = \; & P^{-1}\,\mathrm{d}L - LP^{-2}\,\mathrm{d}P \\ & + LP^{-3}\,\mathrm{d}[P] - P^{-2}\,\mathrm{d}[L,P] \\ = \; & \frac{L}{P}\left[\mu_L - \pi_P + \|\sigma_P\|^2 - \sigma_L^\top\sigma_P\right]\mathrm{d}t \\ & + \frac{L}{P}(\sigma_L - \sigma_P)^\top\,\mathrm{d}W, \end{aligned}$$

which proves (3). The asset derivation in (6) is identical after replacing L by S^i.
The second calculation establishes the risk adjusted human-capital pricing equation.
Under $\mathbb{Q}$, $W^{\mathbb{Q}} = W + \lambda t$, so

$$\mathrm{d}Y = Y(\alpha_Y - \eta^\top\lambda)\,\mathrm{d}t + Y\eta^\top\,\mathrm{d}W^{\mathbb{Q}}.$$

For $H(t,y) = h(t)y$, the Feynman–Kac equation is

$$\begin{aligned} H_t \; & + (\alpha_Y - \eta^\top\lambda)yH_y + 1/2\,y^2\|\eta\|^2 H_{yy} \\ & - rH + y = 0, \qquad H(T,y) = 0. \end{aligned}$$

Because $H_{yy} = 0$, division by y gives (26); solving that linear terminal-value problem yields (25).
The verification argument then connects the smooth HJB candidate to the attainable expected utility.
Let $\hat{V}$ be the smooth candidate in Theorem 6.1. For an arbitrary admissible policy, the HJB inequality implies

$$U(c_s) - \delta\hat{V} + \hat{V}_t + (rZ_s + \tilde{a}_s^\top b - c_s)\hat{V}_z + 1/2\,\tilde{a}_s^\top C\tilde{a}_s\hat{V}_{zz} \le 0.$$

Apply Itô's formula to $e^{-\delta(s-t)}\hat{V}(s,Z_s)$ up to a localising sequence. After integration and expectation, the stochastic integral vanishes and $\hat{V}(t,z) \ge J(c,a)$. Equality holds at the maximisers. Uniform integrability permits passage to the horizon.
The limiting logarithmic specification confirms that the human-capital hedge is not an artefact of a particular CRRA coefficient.
For $\gamma = 1$, take $U(c) = \log c$. The speculative total-wealth portfolio is

$$\tilde{a}^* = ZC^{-1}b,$$

and the financial portfolio again subtracts a^H. Consumption has the form $c^* = Z/F_1(t)$, where

$$F_1(t) = \frac{1-e^{-\delta(T-t)}}{\delta} + \beta e^{-\delta(T-t)}.$$

This limit confirms that the income hedge is not a CRRA artefact.

Table 3: Notation

Symbol	Definition
P, L, Y	price index, nominal labour income and real labour income L/P
S^i, $\tilde{S}^i$	nominal and real price of risky asset i
r	real locally riskless rate
b	vector of real expected excess returns
Σ, C	real-return volatility and covariance matrices, $C = \Sigma\Sigma^T$
λ	market-price-of-risk vector $\Sigma^{-1}b$ in complete markets
αY, η	drift and Brownian exposure of real labour income
σ⊥	unspanned real-income volatility
X, Y, Z	financial wealth, real income and total wealth X + hY
c, a	real consumption and risky-asset amount vector
z, u, p	normalised states and controls X/Y, c/Y, a/Y
γ, δ, β	risk aversion, discounting and bequest weight
h	risk-adjusted human-capital factor
F	consumption annuity factor
V, v	original and scale-reduced value functions
K	feasible portfolio correspondence

References

1. Merton RC. Lifetime portfolio selection under uncertainty: the continuous-time case. Rev Econ Stat. 1969;51(3):247-257. doi:10.2307/1926560.
2. Merton RC. Optimum consumption and portfolio rules in a continuous-time model. J Econ Theory. 1971;3(4):373-413. doi:10.1016/0022-0531(71)90038-X.
3. Samuelson PA. Lifetime portfolio selection by dynamic stochastic programming. Rev Econ Stat. 1969;51(3):239-246. doi:10.2307/1926559.
4. Bodie Z, Merton RC, Samuelson WF. Labor supply flexibility and portfolio choice in a life cycle model. J Econ Dyn Control. 1992;16(3-4):427-449. doi:10.1016/0165-1889(92)90044-F.
5. Viceira LM. Optimal portfolio choice for long-horizon investors with nontradable labor income. J Finance. 2001;56(2):433-470. doi:10.1111/0022-1082.00333.
6. Cocco JF, Gomes FJ, Maenhout PJ. Consumption and portfolio choice over the life cycle. Rev Financ Stud. 2005;18(2):491-533. doi:10.1093/rfs/hhi017.
7. Brennan MJ, Xia Y. Dynamic asset allocation under inflation. J Finance. 2002;57(3):1201-1238. doi:10.1111/1540-6261.00459.
8. Munk C, Sørensen C, Vinther TN. Dynamic asset allocation under mean-reverting returns, stochastic interest rates, and inflation uncertainty: are popular recommendations consistent with rational behavior? Int Rev Econ Finance. 2004;13(2):141-166. doi:10.1016/S1059-0560(03)00051-0.
9. Karatzas I, Lehoczky JP, Sethi SP, Shreve SE. Explicit solution of a general consumption/investment problem. Math Oper Res. 1986;11(2):261-294. doi:10.1287/moor.11.2.261.
10. Cox JC, Huang CF. Optimal consumption and portfolio policies when asset prices follow a diffusion process. J Econ Theory. 1989;49(1):33-83. doi:10.1016/0022-0531(89)90067-7.
11. Karatzas I, Shreve SE. Methods of mathematical finance. New York: Springer; 1998. doi:10.1007/b98840.
12. Pham HY. Continuous-time stochastic control and optimization with financial applications. Berlin: Springer; 2009. doi:10.1007/978-3-540-89500-8.
13. Campbell JY, Viceira LM. Strategic asset allocation: portfolio choice for long-term investors. Oxford: Oxford University Press; 2002.
14. Gomes FJ, Kotlikoff LJ, Viceira LM. Optimal life-cycle investing with flexible labor supply: a welfare analysis of life-cycle funds. Am Econ Rev. 2008;98(2):297-303. doi:10.1257/aer.98.2.297.
15. Zhang A, Ewald C. Optimal investment for a pension fund under inflation risk. Math Methods Oper Res. 2010;71:353-369. doi:10.1007/s00186-009-0294-5.
16. Cairns AJG, Blake D, Dowd K. Stochastic lifestyling: optimal dynamic asset allocation for defined contribution pension plans. J Econ Dyn Control. 2006;30(5):843-877. doi:10.1016/j.jedc.2005.03.009.
17. Crandall MG, Ishii H, Lions PL. User's guide to viscosity solutions of second order partial differential equations. Bull Am Math Soc. 1992;27(1):1-67. doi:10.1090/S0273-0979-1992-00266-5.
18. Fleming WH, Soner HM. Controlled Markov processes and viscosity solutions. 2nd ed. New York: Springer; 2006. doi:10.1007/0-387-31071-1.
19. Deaton A. Saving and liquidity constraints. Econometrica. 1991;59(5):1221-1248. doi:10.2307/2938366.
20. Carroll CD. Buffer-stock saving and the life cycle/permanent income hypothesis. Q J Econ. 1997;112(1):1-55. doi:10.1162/003355397555109.
21. Kushner HJ, Dupuis PG. Numerical methods for stochastic control problems in continuous time. 2nd ed. New York: Springer; 2001. doi:10.1007/978-1-4613-0007-6.
22. Davis MHA, Norman AR. Portfolio selection with transaction costs. Math Oper Res. 1990;15(4):676-713. doi:10.1287/moor.15.4.676.

How to Cite This Article

Nangolo PN. Optimal investment and consumption under inflation and labour income uncertainty. Int J Appl Math Numer Res. 2026;2(4):36-49.
doi:10.54660/IJAMNR.2026.2.4.36-49.

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