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Chaos Suppression Effect of Density-Equicontinuity in Time-Varying Product Systems

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Abstract

This paper explores the intrinsic restrictive relationship between density-equicontinuity of factor systems and two classical chaotic properties within the framework of time-varying discrete product dynamical systems. Under the minimality assumption of product systems, it is proven that the product system admits neither Devaney chaos nor Li-Yorke chaos under the conditions that all its factor systems are density-equicontinuous. Furthermore, the converse proposition is established: the existence of chaotic behavior in a minimal product system necessarily indicates that at least one corresponding factor system fails to possess density-equicontinuity.

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1. Introduction

Equicontinuity is a core concept in topological dynamical systems for characterizing orbit stability over long times. It contrasts sharply with chaos and sensitive dependence on initial conditions, and is widely studied in this field (Gottschalk and Hedlund, 1955) [1], (Good *et al*., 2020) [2]. Classical global equicontinuity is too restrictive to model systems whose orbits only separate at sparse times. To overcome this drawback, Fomin (1951) [3] first proposed the concept of mean stability. Li *et al*. (2015) later developed mean equicontinuity, which describes statistical orbital deviations using Cesàro averages. Based on these works, Li and Tu (2021) defined density-equicontinuity and almost density-equicontinuity. They adopt upper density to measure how sparse the divergence time sets of orbits are, establishing a framework for analyzing statistical stability.

Two core chaos definitions prevail in topological dynamics. Li-Yorke chaos and Devaney chaos. Li and Yorke (1975) established the rigorous mathematical definition of chaos together with the well-known 'period three implies chaos' result, which describes dynamical disorder through uncountable scrambled sets. Later, Devaney (1989) put forward a topological chaos definition relying on transitivity, dense periodic points and sensitive dependence. Banks *et al*. (1992) further verified that the first two conditions inherently imply sensitivity. Huang and Ye (2002) proved that any Devaney chaotic system defined on an infinite compact metric space contains uncountable Li-Yorke scrambled pairs. Shi and Chen (2009) systematically extended the above two classical chaotic frameworks to the setting of time-varying (i.e. non-autonomous) discrete dynamical systems. Although existing studies have discussed the inheritance of chaotic features for time-varying product systems (Pi *et al*., 2023), (Anwar *et al*., 2022) [12], little literature connects density-based statistical stability with the suppression of chaos.

In our previous work(Zhang *et al*., 2026) [13] on autonomous systems, the equivalence of mean equicontinuity, density-t-equicontinuity for all $t \in [0,1)$, and almost density-equicontinuity were proved. Moreover, two results were obtained. That is, minimal density-equicontinuous systems admit no Devaney chaos, while all density-equicontinuous systems have Li-Yorke scrambled pairs with zero upper density.

Autonomous systems cannot model evolution with time-varying perturbations. Time-varying discrete dynamical system (T-VDDSs) were pioneered by Kolyada and Snoha(1996). Existing literature investigates transitivity, Existing literature investigates transitivity, chaoticity, and their product inheritance(Pi *et al*., 2023), (Kolyada and Snoha, 1996}[14], yet research on density-based statistical stability remains insufficient.Time-varying product systems are widely adopted to analyze multi-component non-autonomous evolutions(Wang *et al*., 2012) [15], (Gu *et al*., 2021). The inheritance of topological properties on product spaces is a central topic in T-VDDSs, and most existing works focus on transitivity, shadowing and sensitive dependence on initial conditions(Pi *et al*., 2023), (Anwar *et al*., 2022) [12]. Minimality shapes the global dynamical behaviors of systems. Current research has not explored the mutual restriction between density stability and Devaney, Li-Yorke chaos under product minimality, nor studied whether product systems composed of density-stable factors admit Li-Yorke pairs. Examples constructed via Weyl equidistribution and subadditivity of upper density can supplement the corresponding theoretical proofs(Galatolo Sorrentino, 2022) [17].

This paper centers its investigation on two core questions: How will the density-based equicontinuity of the factor systems constrain the Li-Yorke chaos and Devaney chaos characteristics of the time-varying product system? Conversely, what constraints does the emergence of chaos in a product system impose on the density-based equicontinuous properties of its factor components?

2. Preliminaries

Some density-based equicontinuity and the fundamental notations for the analysis of T-VDDS product systems are presented in this section.

2.1. Upper density of number set.

Let $\mathbb{N}=\{1,2,3,\ldots\}$ denote the set of all positive integers, and $\mathbb{N}_+=\mathbb{N}\cup\{0\}$ represents the collection of non-negative integers. For any subset $S\subseteq\mathbb{N}$, we adopt the following definition of upper density.

Definition 2.1 For an arbitrary set $S\subseteq\mathbb{N}$, the upper density $\bar{D}(S)$ is defined as

$$\bar{D}(S)=\limsup_{n\to\infty}\frac{|S\cap[0,n-1]|}{n}$$

where $|\cdot|$ stands for the cardinality of a set, which counts the total number of elements contained in the corresponding set. These two properties will be used repeatedly in the derivation and proof of product space separation sets in section 3.

First, upper density possesses shift invariance. For any fixed positive integer t, the shifted set

$$S+t=\{k+t\,|\,k\in S\}$$

obeys

$$\bar{D}(S+t)=\bar{D}(S)$$

Second, the subadditive property holds for any subsets $A,B\subseteq\mathbb{N}$. In other words, the upper density of their union is bounded above by the sum of the upper densities of the two individual sets.

$$\bar{D}(A\cup B)\le\bar{D}(A)+\bar{D}(B).$$

2.2. Time-varying discrete dynamical systems (T-VDDSs).

Suppose X denotes a compact metric space endowed with metric d_X, and $\{F_n\}_{n=1}^{\infty}$ stand for a sequence of continuous self-maps defined on X.

The triple $(X,d_X,\{F_n\}_{n=1}^{\infty})$ is named a time-varying discrete dynamical system, abbreviated as T-VDDS. For any $m\in\mathbb{N}_+$ and positive integer k, the k-step composite iteration starting from m is written as

$$F_k^{(m)}=F_{m+k}\circ F_{m+k-1}\circ\cdots\circ F_{m+1}.$$

Specifically, if $m=0$, the iteration starts from the first map in the sequence, denoted by

$$F_1^k=F_1^{(0)}=F_k\circ F_{k-1}\circ\cdots\circ F_1$$

and F_0 is defined as the identity mapping $\mathrm{id}:X\to X$ on space X.

For any initial point $x\in X$, the orbit under $F_k^{(m)}(x)$ is

$$\mathrm{Orb}(x)=\{F_k^{(m)}(x)\,|\,m\in\mathbb{N}_+,k\in\mathbb{N}\}.$$

Given any $\varepsilon>0$, starting moment $m\in\mathbb{N}_+$, a and two distinct points $x_1,x_2\in X$, the orbital separation set for steps with orbit distance exceeding ε is defined as

$$F_{x_1,x_2}^{(m)}=\left\{k\in\mathbb{N}\,|\,d_X(F_k^{(m)}(x_1),F_k^{(m)}(x_2))>\varepsilon\right\}.$$

Minimality serves as a key topological condition for all chaos-inhibition results in this work.

Definition 2.2 A T-VDDS is minimal if every point $x\in X$ has dense orbit in X. Equivalently, the space X contains no non-empty, closed, proper invariant subsets under the mapping sequence $\{F_n\}_{n=1}^{\infty}$.

2.3. Product T-VDDS.

Definition 2.3 Let

$$(X,d_X,\{F_n\}_{n=1}^{\infty})$$

And

$$(Y, d_Y, \{G_n\}_{n=1}^{\infty})$$

be two compact T-VDDSs. The product space is defined as

$$(X \times Y, \rho, \{F_n \times G_n\}_{n=1}^{\infty}),$$

which is endowed with the Chebyshev metric

$$\rho : (X \times Y) \times (X \times Y) \to \mathbb{R}_+, \text{ where}$$

$$\rho((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

for

$$(x_1, y_1), (x_2, y_2) \in X \times Y.$$

Each map $F_n \times G_n$ in the sequence of product maps $\{F_n \times G_n\}_{n=1}^{\infty}$ is defined as

$$(F_n \times G_n)(x, y) = (F_n(x), G_n(y)),\ \forall (x, y) \in X \times Y.$$

Since F_n and G_n are continuous, every product mapping $F_n \times G_n$ is continuous under the product topology induced by ρ.

The k-step iteration of the product system starting from m satisfies the component-wise decomposition

$$(F \times G)_k^{(m)}(x, y) = (F_k^{(m)}(x), G_k^{(m)}(y)),$$

where $F_k^{(m)}$ and $G_k^{(m)}$ are the k-step iterations of $\{F_n\}_{n=1}^{\infty}$ and $\{G_n\}_{n=1}^{\infty}$, respectively. Take two points

$$z_1 = (x_1, y_1),\ z_2 = (x_2, y_2) \in X \times Y.$$

The orbital separation set of the product system reads

$$(F \times G)_{z_1, z_2}^{(m)} = \left\{ k \in \mathbb{N} \,\middle|\, \rho\left((F \times G)_k^{(m)}(z_1), (F \times G)_k^{(m)}(z_2)\right) > \varepsilon \right\}.$$

A straightforward set inclusion follows.

$$(F \times G)_{z_1, z_2}^{(m)} \subseteq F_{x_1, x_2}^{(m)} \cup G_{y_1, y_2}^{(m)}.$$

This inclusion will serve as the core inequality for proving the preservation of density-equicontinuity on product spaces.

2.4. Density-Based Equicontinuity

Density-based equicontinuities are defined as follows.

Definition 2.4 Let $t \in [0,1]$. A T-VDDS $(X, d, \{F_n\}_{n=1}^{\infty})$ is called density-t-equicontinuous if for arbitrary small $\varepsilon > 0$, there exists a corresponding $\delta > 0$. For any $x_1, x_2 \in X : d(x_1, x_2) < \delta$ and every starting time $m \in \mathbb{N}_+$, the upper density of orbital separation set satisfies

$$\bar{D}(F_{x_1, x_2}^{(m)}) \le 1 - t.$$

In the special case $t = 1$, this yields the definition of density-equicontinuity below.

Definition 2.5 A T-VDDS is density-equicontinuous if for any $\varepsilon > 0$, one can select $\delta > 0$ such that, for all $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$ and all $m \in \mathbb{N}_+$ satisfy

$$\bar{D}(F_{x_1, x_2}^{(m)}) = 0.$$

The definition of almost density-equicontinuity relies on the following local stable point notion.

Definition 2.6 A point $x_0 \in X$ is a density-equicontinuous point if for arbitrary $\varepsilon > 0$, there exists a neighborhood $U = B(x_0, \delta)$ of x_0, for $u, v \in U$ with $d(u, v) < \delta$ and any $m \in \mathbb{N}_+$, the upper density of arbitrary separation set $\bar{D}(F_{u,v}^{(m)}) = 0$.

Definition 2.7 A T-VDDS is almost density-equicontinuous if, there exists at least one transitive density-equicontinuous point $x_0 \in X$. Where 'transitive point' refers to a point whose orbit is dense in X.

2.5. Two types of chaotic criteria.

This section concludes with standard definitions of Li-Yorke chaos and Devaney chaos, which characterize complex dynamical behaviors contrary to density-based equi-continuity.

Definition 2.8 A T-VDDS is Li-Yorke chaotic if, there exists an uncountable subset $S \subseteq X$ without any periodic points, for any two distinct elements $x, y \in S$, the orbit distances satisfy

$$\liminf_{k \to \infty} d(F_1^k(x), F_1^k(y)) = 0$$

and

$$\limsup_{k \to \infty} d(F_1^k(x), F_1^k(y)) > 0.$$

The set S is named an uncountable scrambled set for the system.

Definition 2.9 A T-VDDS is Devaney chaotic if three criteria listed below are satisfied.

- **Topological Transitivity**

For any pair of nonempty open subsets $U, V \subset X$, an integer $k \in \mathbb{N}$ exists with

$$F_1^k(U)\cap V\neq\varnothing.$$

- **Dense Asymptotically Periodic Points**

Following the definition of asymptotically periodic points introduced by Pravec (2019) [18], the set of all asymptotically periodic points is dense in X.

- **Sensitive Dependence on Initial Conditions**

There is a constant $\gamma>0$, for any $x\in X$ and any neighborhood $B(x,\delta)$ of x one can pick a point $y\in B(x,\delta)$ and an integer $k\in\square$ satisfying

$$d(F_1^k(x),\ ,F_1^k(y))>\gamma.$$

3. Main results

Let $(X,d_X,\{F_n\}_{n=1}^{\infty})$ and $(Y,d_Y,\{G_n\}_{n=1}^{\infty})$ be T-VDDSs. Their product system is

$$(X\times Y,\rho,\{F_n\times G_n\}_{n=1}^{\infty}),$$

endowed with the Chebyshev metric

$$\rho((x_1,y_1),(x_2,y_2))=\max\{d_X(x_1,x_2),d_Y(y_1,y_2)\},$$

$$\forall(x_1,y_1),(x_2,y_2)\in X\times Y.$$

Proposition 3.1 No minimal T-VDDS equipped with density-equicontinuity can satisfy the Devaney chaos definition.

Proof. Let $(X,d_X,\{F_n\}_{n=1}^{\infty})$ is a minimal T-VDDS. Suppose X is Devaney chaotic with sensitivity constant $\gamma>0$. Density-equicontinuity yields δ_0 such that any $x,y\in X$ with $d_X(x,y)<\delta_0$ possess separation sets of upper density zero. Together with minimality and uniform continuity of finite iterates, $d(F_1^kx,F_1^ky)<\gamma$ holds for all k. This contradicts to sensitivity. Hence X cannot be Devaney chaotic. □

Theorem 3.1
Let

$$(X,d_X,\{F_n\}_{n=1}^{\infty})$$

and

$$(Y,d_Y,\{G_n\}_{n=1}^{\infty})$$

be two T-VDDSs. If each factor system is density-equicontinuous, and the product system

$$(X\times Y,\rho,\ \{F_n\times G_n\}_{n=1}^{\infty})$$

is minimal, then, the product system cannot be Devaney chaotic.

Proof. Fix any $\varepsilon>0$. Since X is density-equicontinuous, then there exists a $\delta_1>0$ such that for any $x_1,x_2\in X$ with

$$d_X(x_1,x_2)<\delta_1$$

and all $m\in\square_+$,

$$\bar{D}(\{k\in\square:d_X(F_k^{(m)}(x_1),F_k^{(m)}(x_2))>\varepsilon\})=0.$$

Similarly, there exists a $\delta_2>0$ for system Y. Set $\delta=$. $\min\{\delta_1,\delta_2\}$. Take any $z_1=(x_1,y_1),z_2=(x_2,y_2)$ $\in X\times Y$ satisfying $\rho(z_1,z_2)<\delta$. one has $d_X(x_1,x_2)$ $<\delta_1$ and $d_Y(y_1,y_2)<\delta_2$. By definition of ρ,

$$\rho((F\times G)_k^{(m)}(z_1),(F\times G)_k^{(m)}(z_2))>\varepsilon$$

holds if and only if

$$d_X(F_k^{(m)}(x_1),F_k^{(m)}(x_2))>\varepsilon \quad \text{or}$$

$$d_Y(G_k^{(m)}(y_1),G_k^{(m)}(y_2))>\varepsilon.$$

Denote

$$F_{x_1,x_2}^{(m)}=\{k\in\square\,|\,d_X(F_k^{(m)}(x_1),F_k^{(m)}(x_2))>\varepsilon\},$$

$$G_{y_1,y_2}^{(m)}=\{k\in\square\,|\,d_Y(G_k^{(m)}(y_1),G_k^{(m)}(y_2))>\varepsilon\},$$

$$(F\times G)_{z_1,z_2}^{(m)}=\{k\in\square\,|\,\rho((F\times G)_k^{(m)}(z_1),(F\times G)_k^{(m)}(z_2))>\varepsilon\}.$$

The inclusion

$$(F\times G)_{z_1,z_2}^{(m)}\subseteq F_{x_1,x_2}^{(m)}\cup G_{y_1,y_2}^{(m)}$$

follows directly. Subadditivity of upper density yields

$$\bar{D}((F\times G)_{z_1,z_2}^{(m)})\leq\bar{D}(F_{x_1,x_2}^{(m)})+\bar{D}(G_{y_1,y_2}^{(m)})=0.$$

Thus $X\times Y$ is density-equicontinuous.
Since $X\times Y$ is minimal and density-equicontinuous, by Proposition 3.1, $X\times Y$ is not Devaney chaotic. □
Remark. The condition 'product minimality' in Theorem 3.1 is sufficient but not necessary. Example 3.1 gives a non-minimal product system whose factor system X lacks topological transitivity, rendering the product non-Devaney chaotic. Besides, minimality of individual factor system does not ensure minimality of the product system $X\times Y$, so minimality must be required for the full product space itself.

Example 3.1 Equip X with Euclidean metric $d_X(s_1, s_2) = |s_1 - s_2|$. Set $F_n(s) = \frac{s}{2}$ for all $n \in \mathbb{N}$. Direct computation yields

$$F_k^{(m)}(s) = \frac{s}{2^k}.$$

Take arbitrary $\varepsilon > 0$ and choose $\delta_X = \varepsilon$. For $s_1, s_2 \in X$ satisfying

$$d_X(s_1, s_2) < \delta_X,$$

$$d_X(F_k^{(m)}(s_1), F_k^{(m)}(s_2)) = \frac{|s_1 - s_2|}{2^k} < \varepsilon$$

holds for all $m \in \mathbb{Z}_+, k \in \mathbb{N}$. This yields the orbital separation set

$$F_{s_1,s_2}^{(m)} = \{k \in \mathbb{N} \mid d_X(F_k^{(m)}(s_1), F_k^{(m)}(s_2)) > \varepsilon\} = \varnothing,$$

and thus its upper density satisfies

$$\bar{D}(F_{s_1,s_2}^{(m)}) = 0.$$

Hence, X is density-equicontinuous
Equip the unit circle $Y = \mathbb{R}/\mathbb{Z}$ with circular metric

$$d_Y(t_1, t_2) = \min\{|t_1 - t_2|, 1 - |t_2 - t_1|\}.$$

Fix an irrational number $\alpha \in (0,1)$ and define

$$G_n(t) = (t + \alpha) (\mathrm{mod}\, 1).$$

The k-step iteration reads

$$G_k^{(m)}(t) = (t + k\alpha) (\mathrm{mod}\, 1).$$

Since the orbit

$$\{t + k\alpha\, (\mathrm{mod}\, 1) \mid k \in \mathbb{N}\}$$

of every point is dense in T, Y is minimal.
Circle rotations preserve the circular metric uniformly, which is a fundamental isometry property detailed in (Katok and Hasselblatt, 1995) [20], so

$$d_Y(G_k^{(m)}(t_1), G_k^{(m)}(t_2)) = d_Y(t_1, t_2)$$

for all

$$m \in \mathbb{Z}_+, k \in \mathbb{N}.$$

Pick

$$\delta_Y = \varepsilon.$$ If $d_Y(t_1, t_2) < \delta_Y$

then every iterate satisfies

$$d_Y(G_k^{(m)}(t_1), G_k^{(m)}(t_2)) < \varepsilon.$$

Thus $G_{t_1,t_2}^{(m)} = \varnothing$ and $\bar{D}(G_{t_1,t_2}^{(m)}) = 0$. Hence, Y is density-equicontinuous.
Given $\varepsilon > 0$, let

$$\delta = \min\{\delta_X, \delta_Y\}.$$

Take arbitrary

$$z_1 = (s_1, t_1),\ z_2 = (s_2, t_2) \in X \times Y$$

with

$$\rho(z_1, z_2) < \delta.$$

Then,

$$d_X(s_1, s_2) < \delta_X$$

and

$$d_Y(t_1, t_2) < \delta_Y.$$

The Chebyshev metric implies

$$(F \times G)_{z_1,z_2}^{(m)} = F_{s_1,s_2}^{(m)} \cup G_{t_1,t_2}^{(m)}.$$

Applying subadditivity of upper density, one has

$$\bar{D}((F \times G)_{z_1,z_2}^{(m)}) \le \bar{D}(F_{s_1,s_2}^{(m)}) + \bar{D}(G_{t_1,t_2}^{(m)}) = 0.$$

So, $X \times Y$ is density-equicontinuous.
While, select open subsets

$$U = (0.5, 1],\ V = (0.9, 1) \subset X.$$

For any $m \in \mathbb{Z}_+$, $k \in \mathbb{N}$ one has

$$F_k^{(m)}(U) \subseteq [0,\ 0.5],$$

which gives

$$F_k^{(m)}(U) \cap V = \varnothing.$$

The corresponding product open sets satisfy

$$(F \times G)_k^{(m)}(U \times Y) \cap (V \times Y) = \varnothing$$

for all $m \in \mathbb{Z}_+$, $k \in \mathbb{N}$. Topological transitivity fails. So, $X \times Y$ cannot be Devaney chaotic.

This example shows that the product of two density-equicontinuous systems still carries density-equicontinuity, even though the product system fails minimality and cannot be Devaney chaotic.

Proposition 3.2 Let

$$(X, d_X, \{F_n\}_{n=1}^{\infty})$$

and

$$(Y, d_Y, \ \{G_n\}_{n=1}^{\infty})$$

be two T-VDDSs. If the product system is minimal, then both factor systems are minimal.

Proof. For a T-VDDS, minimality is equivalent to every point having dense orbit. Assume $X \times Y$ is minimal. Take arbitrary $x \in X$ and a fixed $y_0 \in Y$, then (x, y_0) is a transitive point of the product system. By fundamental properties of continuous projections in general topology (Kelley, 2017) [19], continuous maps commute with set closures, then

$$\overline{\mathrm{Orb}(x)} = \overline{\pi_X(\mathrm{Orb}(x, y_0))} = \pi_X(\overline{\mathrm{Orb}(x, y_0)})$$

$$= \pi_X(X, Y) = X.$$

Since x is arbitrary, one obtains that X is minimal. The same reasoning applies to the system Y. □

Remark. Minimality of both factor systems do not guarantee minimality of the product system.

Corollary 3.1 Let X, Y be T-VDDSs and $X \times Y$ is minimal. If $X \times Y$ is Devaney chaotic, then at least one of X, Y fails density-equicontinuous.

Proof. Assume conversely that $X \times Y$ is minimal and Devaney chaotic, and both X, Y are density-equicontinuous. By Theorem 3.1, the product system is density-equicontinuous. Combined with minimality and Proposition 3.1, $X \times Y$ cannot be Devaney chaotic. This contradiction completes the proof. □

Corollary 3.2 Let X, Y be T-VDDS and $X \times Y$ is minimal. If X is Devaney chaotic, then X is not density-equicontinuous.

Proof. Since $X \times Y$ is minimal, Proposition 3.2 derives that X is minimal. If X is density-equicontinuous, Proposition 3.1 ensures that X is free of Devaney chaos, which contradicts that X is Devaney chaotic. Hence, X is not density-equicontinuous. □

Theorem 3.2 Let $(X, d, \{F_n\}_{n=1}^{\infty})$ be a density-equicontinuous T-VDDS. Denote

$$LY = \Big\{(x, y) \in X \times X \,\Big|\, \liminf_{k \to \infty} d(F_1^k(x), F_1^k(y)) = 0,$$

$$\limsup_{k \to \infty} d(F_1^k(x), F_1^k(y)) > 0\Big\}.$$

For any

$(x, y) \in LY$, write

$$\eta = \limsup_{k \to \infty} d(F_1^k(x), F_1^k(y))$$

and

$$G = \{n \in \mathbb{N} \,|\, d(F_1^n(x), F_1^n(y)) > \eta\}.$$

Then,

$$\bar{D}(G) = 0.$$

Proof. By the definition of the upper limit, G is infinite.

Fix $\varepsilon = \dfrac{\eta}{2}$. By density-equicontinuity of the system, there exists a $\delta > 0$ such that for $u, v \in X$ with $d(u, v) < \delta$ and all $m \in \mathbb{Z}_+$,

$$\bar{D}(\{s \in \mathbb{N} \,|\, d(F_s^{(m)}(u), F_s^{(m)}(v)) > \varepsilon\}) = 0.$$

Since

$$\liminf_{k \to \infty} d(F_1^k(x), F_1^k(y)) = 0,$$

pick $k_0 \in \mathbb{N}$ and set

$$u = F_1^{k_0}(x), \ v = F_1^{k_0}(y),$$

one has

$d(u, v) < \delta$. Let

$$S = \{n \in \mathbb{N} \,|\, d(F_1^n(x), F_1^n(y)) > \varepsilon\},$$

$$S' = \{s \in \mathbb{N} \,|\, d(F_s^{(k_0)}(u), F_s^{(k_0)}(v)) > \varepsilon\}$$

$$= \{s \in \mathbb{N} \,|\, d(F_1^{k_0+s}(x), F_1^{k_0+s}(y)) > \varepsilon\}.$$

Iteration composition yields

$$S' = \{s \in \mathbb{N} \,|\, k_0 + s \in S\}.$$

Shift invariance of upper density gives

$$\bar{D}(S) = \bar{D}(S') = 0.$$

Since $G \subseteq S$, monotonicity of upper density yields

$$\bar{D}(G) \le \bar{D}(S) = 0.$$

Hence, $\bar{D}(G) = 0$. □

Corollary 3.3 Let

$$(X, d, \{F_n\}_{n=1}^{\infty})$$

be a minimal density-equicontinuous T-VDDS. Then X contains no Li-Yorke pairs, hence the system is not Li-Yorke chaotic.

Proof. Assume conversely that there exists a Li-Yorke pair

$$(x, y) \in LY.$$

Set

$$\eta = \limsup_{k\to\infty} d(F_1^k x, F_1^k y) > 0$$

and

$$G = \{n \in \mathbb{N} \,|\, d(F_1^n x, F_1^n y) > \eta\}.$$

By Theorem 3.2, $\bar{D}(G) = 0$, which means large separations above η occur only on a zero-density set of iterations.

Since the system is minimal, the orbit of (x, y) is dense in $X \times X$, together with zero upper density separation sets

$$\bar{D}(G) = 0,$$

yield

$$\limsup_{k\to\infty} d(F_1^k x, F_1^k y) = 0$$

for any

$$(x, y) \in LY.$$

This contradicts to $\eta > 0$. Hence, no Li-Yorke pairs exist in X. □

Theorem 3.3 Let X, Y be T-VDDSs, and the product system $X \times Y$ is minimal. If $X \times Y$ admits a Li-Yorke pair, then at least one of X, Y is not density-equicontinuous.

Proof. Assume conversely that both X and Y are density-equicontinuous.

By Theorem 3.1, the product system $X \times Y$ is also density-equicontinuous.

Since $X \times Y$ is minimal, Proposition 3.2 yields that the factor systems X and Y are both minimal.

Then, $X \times Y$ is a minimal density-equicontinuous T-VDDS.

Corollary 3.3 guarantees that $X \times Y$ possesses no Li-Yorke scrambled pairs.

This contradicts to the existence of a Li-Yorke pair in $X \times Y$. Therefore, at least one of X, Y is not density-equicontinuous. □

Corollary 3.4 Let X, Y be T-VDDSs and $X \times Y$ is minimal. If both X and Y are density-equicontinuous, then there exists no Li-Yorke pair in $X \times Y$.

Proof. This is the contrapositive statement of Theorem 3.3.

Corollary 3.5 Let X, Y be minimal, density-equicontinuous T-VDDSs. If the product system $X \times Y$ is minimal, then $X \times Y$ is neither Devaney chaotic nor Li-Yorke chaotic.

Proof. Since X, Y are density-equicontinuous and $X \times Y$ is minimal, Theorem 3.1 ensures that $X \times Y$ cannot be Devaney chaotic. Meanwhile, Corollary 3.4 yields no Li-Yorke pairs exist in $X \times Y$, so $X \times Y$ is not Li-Yorke chaotic. □

Remark. All preceding theorems rely on two key assumptions. The one is that both factors are density-equicontinuous, another is that the product system is minimal. It is natural to investigate mixed product systems consisting of a density-equicontinuous factor system and a almost density-equicontinuous factor system along with no product minimality. Example 3.6 below constructs such a non-minimal product system and demonstrates that such mixed settings exhibit complex chaos behaviors.

Example 3.6 Let $X = [0,1]$ be equipped with Euclidean metric d_X. Define the constant mapping sequence $F_n(x) = \dfrac{x}{2}$ for all $n \in \mathbb{N}$. As verified in Example 3.1, take $\delta = \varepsilon$ for any $\varepsilon > 0$. For any

$$x_1, x_2 \in X$$ with

$$d_X(x_1, x_2) < \delta,$$

the separation set $F_{x_1,x_2}^{(m)}$ is empty and $\bar{D}(F_{x_1,x_2}^{(m)}) = 0$. The T-VDDS X satisfies Definition 2.5 and is density-equicontinuous.

Let $Y = \mathrm{T} = \mathbb{R}/\mathbb{Z}$ denote the unit circle, also denoted by S^1 (Galatolo and Sorrentino, 2022) [17]. The following identifies this space with the half-open interval $[0,1)$, where the endpoints 0 and 1 represent the same point, and equip it with the standard circular metric.

$$d_Y(y_1, y_2) = \min\{|y_1 - y_2|, 1 - |y_1 - y_2|\}.$$

Fix an irrational number

$$\beta \in (0, 0.5).$$

For each $n \in \mathbb{N}$, the continuous self map $G_n : Y \to Y$ is defined piecewise by

$$G_n(t)=\begin{cases}(y+\beta)\ (\mathrm{mod}\ 1), & y\in[0,0.5],\\ 0.5+\beta-2\beta(y-0.5), & y\in(0.5,1).\end{cases}$$

The translation on $[0,0.5]$ and linear function on $(0.5,1)$ are continuous on their respective open intervals. The left and right limits at $y=0.5$ both equal $0.5+\beta$, so G_n is continuous on the whole circle. Thus, $(Y,d_Y,\{G_n\}_{n\in\mathbb{N}})$ forms a T-VDDS. In what follows, we verify the main dynamical properties of Y sequentially.

Take any point $y_0\in(0,0.5)$. The orbit $\{(y_0+k\beta)\ (\mathrm{mod}\,1)\,|\,k\in\mathbb{N}\}$ is dense on T. This classical property of irrational rotations is widely adopted (Galatolo and Sorrentino, 2022) [17], which implies y_0 is a transitive point of Y.

Given arbitrary $\varepsilon>0$, select a neighborhood $U\subset(0,0.5)$ of t_0 with diameter smaller than ε. For any $y_1,y_2\in U$ satisfying $d_Y(y_1,y_2)<\varepsilon$, circular distances are preserved under irrational translation

$$d_Y(G_k^{(m)}(y_1),G_k^{(m)}(y_2))=d_Y(y_1,y_2),$$

$$\forall m\in\mathbb{N}_+,k\in\mathbb{N}.$$

Hence, Y is almost density-equicontinuous.

Pick

$$y_1=0,y_2=0.5\in[0,0.5]$$

and fix $\varepsilon_0=0.05$. For any $m\in\mathbb{N}_+,k\in\mathbb{N}$,

$$G_k^{(m)}(y_1)=k\beta(\mathrm{mod}1),$$

$$G_k^{(m)}(y_2)=0.5+\beta.$$

The separation set reads

$$G_{0,0.5}^{(m)}=\{k\in\mathbb{N}\,|\,d_Y(k\beta(\mathrm{mod}1),0.5+\beta)>0.05\}.$$

By Weyl's equidistribution theorem (Katok and Hasselblatt, 1995) [20], $\{k\beta(\mathrm{mod}1)\}$ is uniformly distributed on T. Let

$$I=\{y\in\mathrm{T}\,|\,d_Y(y,0.5+\beta)>0.05\},$$

whose Lebesgue length

$$|I|=1-2\times0.05=0.9>0.$$

Uniform distribution implies

$$\lim_{N\to\infty}\frac{1}{N}\#\{1\le k\le N\,|\,k\beta(\mathrm{mod}1)\in I\}=0.9,$$

So,

$$\bar{D}(G_{0,0.5}^{(m)})=0.9>0.$$

This violates density-equicontinuity, which requires $\bar{D}(G_{y_1,y_2}^{(m)})=0$ for any nearby points $y_1,y_2\in Y$.

Solve the fixed-point equation $G_n(y)=y$ over $(0.5,1)$,

$$0.5+\beta-2\beta(y-0.5)=y$$

Then,

$$y^*=\frac{0.5+2\beta}{1+2\beta}.$$

Since $0<\beta<0.5$, the root y^* lies in $(0.5,0.75)\subset(0.5,1)$. The orbit of y^* is the singleton y^* and cannot be dense on T. Hence, Y is non-minimal.

Consider the product system $X\times Y$ equipped with Chebyshev metric ρ, choose two points

$$z_1=(0,0),\quad z_2=(1,0.5)$$

from $X\times Y$, for each $k\in\mathbb{N}$, one has

$$(F\times G)_k^{(m)}(z_1)=(0,k\beta\,(\mathrm{mod}\,1)),$$

$$(F\times G)_k^{(m)}(z_2)=\left(\frac{1}{2^k},0.5+\beta\right).$$

The product metric yields

$$\rho((F\times G)_k^{(m)}(z_1),(F\times G)_k^{(m)}(z_2))$$

$$=\max\left\{\frac{1}{2^k},d_Y(k\beta(\mathrm{mod}\,1),0.5+\beta)\right\}.$$

As $k\to\infty$, the term $\frac{1}{2^k}$ converges to zero. Density of

$$\{k\beta(\mathrm{mod}1)\}$$

on T guarantees a subsequence $\{k_j\}_{j\in\mathbb{N}}$ satisfy

$$d_Y(k_j\beta(\mathrm{mod}1),0.5+\beta)\to0.$$

This leads to

$$\liminf_{k\to\infty} \rho((F\times G)_k^{(m)}(z_1),(F\times G)_k^{(m)}(z_2))=0.$$

The next verifies the upper limit is strictly positive by contradiction. Suppose for every $\delta>0$, only finitely many integers k satisfy

$$d_Y(k\beta(\text{mod}1),0.5+\beta)\geq\delta.$$

For sufficiently large k,

$$d_Y(k\beta(\text{mod}1),0.5+\beta)<\delta$$

holds for arbitrarily small δ. Taking $\delta\to 0$, then

$$k\beta(\text{mod}1)\to 0.5+\beta$$

as $k\to\infty$. This contradicts to the density of irrational rotation orbits, which cannot converge to a single point. Therefore, there exists a constant $\delta_0>0$ and an infinite subsequence $\{k_j'\}_{j\in\mathbb{N}}$ with

$$d_Y(k_j'\beta(\text{mod}1),0.5+\beta)\geq\delta_0.$$

one thus obtains

$$\limsup_{k\to\infty} \rho((F\times G)_k^{(m)}(z_1),(F\times G)_k^{(m)}(z_2))\geq\delta_0>0.$$

By Definition 2.6, the pair (z_1,z_2) forms a Li-Yorke pair in $X\times Y$.

This construction only allows the existence of Li-Yorke chaos for the product system. While, the fixed point $y^*\in(0.5,1)$ breaks topological transitivity on both Y and $X\times Y$. So, the product cannot be Devaney chaotic.

4. Conclusions and Further Discussions

This paper establishes the mutual restrictive mechanism between density statistical stability and chaos for minimal time-varying product systems. If a product system induced by two density-equicontinuous factor systems is minimal, it inherits density-equicontinuity and excludes both Devaney and Li-Yorke chaos. Conversely, any minimal product system possessing Li-Yorke pairs or Devaney chaos must contain at least one factor system without density-equicontinuity. Examples verify that a product system, induced by a density-equicontinuous and an almost density-equicontinuous factor systems, may has Li-Yorke pairs, but not Devaney chaos. There are some further research directions, for example, under the minimality assumption of a time-varying system, can we establish relationships between various density-based equicontinuity? What is the chaos criterion for equicontinuous systems?

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