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Stochastic Mittag–Leffler Stability and Numerical Analysis of Variable-Order Fractional Langevin Equations with Generalized (ϱ, χ)-Caputo Derivatives

Dr. S Suresh [1*], M Karthik [2]

[1] Associate Professor, Department of Mathematics, Kongu Arts and Science College, Erode, Tamil Nadu, India

[2] Assistant Professor, Department of Mathematics, KPR College of Arts Science and Research, Coimbatore, Tamil Nadu, India

* Corresponding Author: **Dr. S Suresh**

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Abstract

This paper investigates a new class of generalized variable-order stochastic fractional Langevin equations involving the (ϱ, χ)–Caputo fractional derivative with Mittag–Leffler type non-singular kernel. The proposed model extends classical fractional Langevin systems by incorporating adaptive memory effects through variable-order fractional operators and generalized kernel structures. Using fixed-point techniques, including the Banach contraction principle and Krasnoselskii's theorem, sufficient conditions for the existence and uniqueness of solutions are established. Further, stochastic Mittag–Leffler stability of the considered system is analyzed with the aid of generalized fractional Gronwall inequalities. In addition, several qualitative properties associated with stochastic perturbations, adaptive memory behavior, and nonlocal effects are discussed. To demonstrate the applicability of the theoretical results, an illustrative example together with extensive numerical comparison analysis is presented. Various computational schemes such as finite difference, predictor–corrector, spectral, hybrid operational matrix, and adaptive wavelet-based methods are compared in terms of convergence rate, computational efficiency, stability index, synchronization accuracy, and numerical error behavior. The obtained results show that the proposed generalized variable-order stochastic fractional Langevin model provides improved stability, lower computational error, enhanced adaptive memory characteristics, and superior numerical performance compared with existing fractional Langevin formulations. The developed framework offers a robust mathematical tool for modeling complex stochastic dynamical systems with variable memory and nonlocal interactions arising in engineering, physics, and applied sciences.

Keywords: Variable-order fractional calculus, Stochastic Langevin equations, Mittag–Leffler stability, Adaptive memory, Fixed-point theory

1. Introduction

Fractional calculus has proven to be a powerful mathematical tool for the description of hereditary phenomena, memory dependent processes, anomalous diffusion, viscoelasticity, stochastic dynamics, and complex systems in engineering, physics, biology and control theory. In recent years, fractional differential equations with generalized fractional operators have been extensively studied because of their ability to model more accurately nonlocal interactions and adaptive memory effects than classical integer-order differential equations [1, 3, 5, 10]. The fractional Langevin equation is a fundamental example of fractional differential equations in various fractional dynamical system models, such as models of Brownian motion, stochastic oscillations, thermal fluctuations, viscoelastic systems and random diffusion processes [13, 14]. Theoretical and numerical study of classical Langevin-type models with Riemann–Liouville and Caputo fractional derivatives has been widely explored. But conventional operators for a single kernel have proven to be inadequate to describe adaptive memory systems and variable hereditary behaviors occurring in complex, real-world systems [12, 4]. To address these drawbacks, several generalized fractional operators with non-singular kernel have been proposed such as Caputo–Fabrizio [1], Atangana–Baleanu [2], ϕ-Caputo [6], χ- Hilfer [15] and generalized Liouville–Caputo derivative [16]. These operators offer greater mathematical flexibility and offer better physical interpretation for systems with long-memory and nonlocal interactions. In addition, recently the variable-order fractional operators have attracted considerable attention since the fractional order may be time-varying, space-varying, or state-varying, thus providing a more realistic description of adaptive memory effects [18, 23]. For a while now, there have been multiple authors who have investigated the question of existence and uniqueness, in [7, 8, 9, 20] they studied the fractional Langevin equations with various generalized fractional operators and nonlocal boundary conditions properties of stability, controllability, and numerical analysis. A particular class of fractional Langevin systems with nonlocal and multipoint boundary conditions have exhibited excellent applicability to stochastic control systems, signal processing, viscoelastic mechanics, and anomalous transport phenomena, among others. Furthermore, the influence of stochastic perturbations and delay effects on the qualitative behavior of fractional dynamical systems is significant and cannot be ignored, hence, it is necessary to have a rigorous mathematical analysis of it. Inspired by the above developments, this paper aims to study a new class of generalized variable-order stochastic fractional Langevin equations with a modified Mittag–Leffler kernel of the (ϱ,χ)-Caputo fractional derivative. The proposed model is an extension of some of the current fractional Langevin systems, including memory behavior of variable order as well as stochastic perturbations and generalized kernel structures. Singular and nonsingular fractional kernels are special cases of the considered kernel. The key findings of this work are summarized below: A new generalized (ϱ,χ)-Caputo stochastic fractional Langevin model involving Mittag–Leffler kernel is proposed. The existence of the solutions is determined by Krasnoselskii's: fixed point theorem and the generalized contraction techniques [21, 22]. The Banach contraction mapping principle is used to derive the uniqueness of solution under appropriate Lipschitz-type conditions. The stochastic Mittag–Leffler stability results are studied by means of generalized fractional Gronwall inequalities and variable-order kernel estimates. Several fractional models, such as singular-kernel, nonsingular kernel, delay, stochastic, coupled and adaptive-memory models are developed for numerical comparison analysis. systems. To showcase the efficiency of the proposed approach, an advanced computational comparison is compared among predictor–corrector numerical schemes, spectral numerical schemes, operational matrix numerical schemes, hybrid spectral numerical schemes and wavelet numerical schemes. The results obtained are a generalization and improvement of several existing results from the literature [7, 8, 16, 19, 25]. Moreover, the moving average model with the generalized stochastic variable- order framework offers a more realistic mathematical framework for the modelling of adaptive memory systems with stochastic fluctuations and nonlocal effects emerging in contemporary applied problems. Science and engineering applications. The rest of this paper is organized as follows: In Section II the basic definitions, the generalized variable order fractional operators, Mittag–Leffler kernels and some auxiliary lemmas are introduced, which will be needed in the later analysis. In Section III, the existence and uniqueness results of the proposed generalized variable-order stochastic fractional Langevin equation are discussed using fixed-point approach and contraction principles. Stochastic Mittag–Leffler stability analysis is set up based on the generalized fractional Gronwall-type inequalities and variable-order kernel estimates in Section IV. In Section V, illustrative examples are presented along with extensive numerical comparison analyses with the singular, nonsingular, delay, stochastic, coupled and adaptive-memory fractional models. The model is then compared with various computational scheme(s) such as predictor–corrector, spectral, operational matrix, hybrid spectral, and wavelet, using graphical and numerical simulations, to show the effectiveness and applicability of the proposed model.

2 Auxiliary Results

Here we provide some basic results and definitions needed for the analysis of the proposed variable order fractional Langevin system.

Definition 2.1 *Let $S=[a,T]\subset R$, $\varrho\in C^1(S,R)$ be a strictly increasing function satisfying $\varrho'(\vartheta)\neq 0$ for all $\vartheta\in S$, and let $\alpha(\vartheta)>0$. The generalized variable-order ϱ–Riemann–Liouville fractional integral of a continuous function $z\colon S\to R$ is defined by*

$$I_{a+}^{\alpha(\vartheta);\varrho}z(\vartheta)=\frac{1}{\Gamma(\alpha(\vartheta))}\int_a^{\vartheta}\varrho'(s)(\varrho(\vartheta)-\varrho(s))^{\alpha(\vartheta)-1}z(s)\,ds.$$

Definition 2.2 *Let $\alpha(\vartheta)\in(0,1)$ and let $\chi\colon S\to R$ be a positive continuous function. Assume that $\varrho\in C^1(S,R)$ is a strictly increasing function satisfying $\varrho'(\vartheta)\neq 0$ for all $\vartheta\in S$. The generalized variable-order (ϱ,χ)–Caputo fractional derivative of a function $z\colon S\to\mathbb{R}$ is defined by*

$${}^{C}D_{a+}^{\alpha(\vartheta);\varrho,\chi}z(\vartheta)=\frac{1}{\Gamma(1-\alpha(\vartheta))\chi(\vartheta)}\int_a^{\vartheta}\varrho'(s)(\varrho(\vartheta)-\varrho(s))^{-\alpha(\vartheta)}\frac{d}{ds}\big(\chi(s)z(s)\big)\,ds.$$

Lemma 2.1. *Let $\alpha(\vartheta),\beta(\vartheta)>0$ be variable-order functions and let $z\in C(S,R)$. Then the generalized variable-order ϱ–fractional integral operators satisfy*

$$I_{a^+}^{\alpha(\vartheta);\varrho}I_{a^+}^{\beta(\vartheta);\varrho}z(\vartheta)=I_{a^+}^{\alpha(\vartheta)+\beta(\vartheta);\varrho}z(\vartheta),\qquad\forall\,\vartheta\in S.$$

Proof. By Definition 2.1, we have

$$I_{a^+}^{\alpha(\vartheta);\varrho}I_{a^+}^{\beta(\vartheta);\varrho}z(\vartheta)=\frac{1}{\Gamma(\alpha(\vartheta))}\int_a^{\vartheta}\varrho'(\tau)(\varrho(\vartheta)-\varrho(\tau))^{\alpha(\vartheta)-1}I_{a^+}^{\beta(\vartheta);\varrho}z(\tau)\,d\tau.$$

Substituting the definition of $I_{a^+}^{\beta(\vartheta);\varrho}$, we obtain

$$=\frac{1}{\Gamma(\alpha(\vartheta))\Gamma(\beta(\vartheta))}\int_a^{\vartheta}\int_a^{\tau}\varrho'(\tau)\varrho'(s)(\varrho(\vartheta)-\varrho(\tau))^{\alpha(\vartheta)-1}(\varrho(\tau)-\varrho(s))^{\beta(\vartheta)-1}z(s)\,ds\,d\tau.$$

Changing the order of integration yields

$$= \frac{1}{\Gamma(\alpha(\vartheta))\Gamma(\beta(\vartheta))} \int_a^{\vartheta} \varrho'(s) z(s) [\int_s^{\vartheta} \varrho'(\tau)(\varrho(\vartheta) - \varrho(\tau))^{\alpha(\vartheta)-1} (\varrho(\tau) - \varrho(s))^{\beta(\vartheta)-1} d\tau] ds.$$

Let

$$u = \frac{\varrho(\tau)-\varrho(s)}{\varrho(\vartheta)-\varrho(s)}.$$

Then

$$d\tau = \frac{\varrho(\vartheta)-\varrho(s)}{\varrho'(\tau)} du.$$

Using the Beta-function identity

$$B(x,y) = \int_0^1 u^{x-1} (1-u)^{y-1} du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

we obtain

$$I_{a^+}^{\alpha(\vartheta);\varrho} I_{a^+}^{\beta(\vartheta);\varrho} z(\vartheta) = \frac{1}{\Gamma(\alpha(\vartheta)+\beta(\vartheta))} \int_a^{\vartheta} \varrho'(s)(\varrho(\vartheta) - \varrho(s))^{\alpha(\vartheta)+\beta(\vartheta)-1} z(s)\, ds.$$

Therefore,

$$I_{a^+}^{\alpha(\vartheta);\varrho} I_{a^+}^{\beta(\vartheta);\varrho} z(\vartheta) = I_{a^+}^{\alpha(\vartheta)+\beta(\vartheta);\varrho} z(\vartheta).$$

Hence, the semigroup property of the generalized variable-order ϱ–fractional integral operator holds. This completes the proof. ☐

Lemma 2.2. *Let $z \in C^n(S,R)$ and let $\alpha(\vartheta) \in (n-1,n)$, where $n \in N$. Then the generalized variable-order (ϱ,χ)–Caputo fractional derivative satisfies*

$$I_{a^+}^{\alpha(\vartheta);\varrho,\chi} \left({}^{C}D_{a^+}^{\alpha(\vartheta);\varrho,\chi} z(\vartheta) \right) = z(\vartheta) - \sum_{k=0}^{n-1} \frac{z_{\varrho}^{[k]}(a)}{k!} (\varrho(\vartheta) - \varrho(a))^k.$$

Proof. From the definition of the generalized variable-order (ϱ,χ)–Caputo fractional derivative, we have

$${}^{C}D_{a^+}^{\alpha(\vartheta);\varrho,\chi} z(\vartheta) = I_{a^+}^{n-\alpha(\vartheta);\varrho,\chi} z_{\varrho}^{[n]}(\vartheta).$$

where

$$z_{\varrho}^{[n]}(\vartheta) = \left(\frac{1}{\varrho'(\vartheta)} \frac{d}{d\vartheta} \right)^n z(\vartheta).$$

Applying the operator $I_{a^+}^{\alpha(\vartheta);\varrho,\chi}$ to both sides yields

$$I_{a^+}^{\alpha(\vartheta);\varrho,\chi} \left({}^{C}D_{a^+}^{\alpha(\vartheta);\varrho,\chi} z(\vartheta) \right) = I_{a^+}^{\alpha(\vartheta);\varrho,\chi} I_{a^+}^{n-\alpha(\vartheta);\varrho,\chi} z_{\varrho}^{[n]}(\vartheta).$$

Using the semigroup property of the generalized variable-order fractional integral operator, we obtain

$$I_{a^+}^{\alpha(\vartheta);\varrho,\chi} \left({}^{C}D_{a^+}^{\alpha(\vartheta);\varrho,\chi} z(\vartheta) \right) = I_{a^+}^{n;\varrho,\chi} z_{\varrho}^{[n]}(\vartheta).$$

Applying the generalized Taylor expansion with respect to ϱ, it follows that

$$I_{a^+}^{n;\varrho,\chi} z_{\varrho}^{[n]}(\vartheta) = z(\vartheta) - \sum_{k=0}^{n-1} \frac{z_{\varrho}^{[k]}(a)}{k!} (\varrho(\vartheta) - \varrho(a))^k.$$

Hence,

$$I_{a^+}^{\alpha(\vartheta);\varrho,\chi} \left({}^{C}D_{a^+}^{\alpha(\vartheta);\varrho,\chi} z(\vartheta) \right) = z(\vartheta) - \sum_{k=0}^{n-1} \frac{z_{\varrho}^{[k]}(a)}{k!} (\varrho(\vartheta) - \varrho(a))^k$$

This completes the proof. ☐

Lemma 2.3. Let $\alpha(\vartheta) \geq 0$, $\beta(\vartheta) > 0$, and $\vartheta > a$. Then the following properties hold.

1. The generalized variable-order ϱ–fractional integral satisfies

$$I_{a^+}^{\alpha(\vartheta);\varrho} (\varrho(\vartheta) - \varrho(a))^{\beta(\vartheta)-1} = \frac{\Gamma(\beta(\vartheta))}{\Gamma(\alpha(\vartheta)+\beta(\vartheta))} (\varrho(\vartheta) - \varrho(a))^{\alpha(\vartheta)+\beta(\vartheta)-1}.$$

2. The generalized variable-order (ϱ, χ)–Caputo fractional derivative satisfies

$${}^{C}D_{a^+}^{\alpha(\vartheta);\varrho,\chi} (\varrho(\vartheta) - \varrho(a))^{\beta(\vartheta)-1} = \frac{\Gamma(\beta(\vartheta))}{\Gamma(\beta(\vartheta)-\alpha(\vartheta))} (\varrho(\vartheta) - \varrho(a))^{\beta(\vartheta)-\alpha(\vartheta)-1}.$$

3. *For every $p \in \{0,1,\dots,k-1\}$, $k \in N$, one has*

$${}^{C}D_{a^+}^{\alpha(\vartheta);\varrho,\chi} (\varrho(\vartheta) - \varrho(a))^p = 0.$$

Proof. Using the definition of the generalized variable-order ϱ–fractional integral operator, we obtain

$$I_{a^+}^{\alpha(\vartheta);\varrho} (\varrho(\vartheta) - \varrho(a))^{\beta(\vartheta)-1} = \frac{1}{\Gamma(\alpha(\vartheta))} \int_a^{\vartheta} \varrho'(s)(\varrho(\vartheta) - \varrho(s))^{\alpha(\vartheta)-1} (\varrho(s) - \varrho(a))^{\beta(\vartheta)-1} ds.$$

Introducing the change of variable

$$u = \frac{\varrho(s)-\varrho(a)}{\varrho(\vartheta)-\varrho(a)},$$

the integral becomes

$$= \frac{(\varrho(\vartheta)-\varrho(a))^{\alpha(\vartheta)+\beta(\vartheta)-1}}{\Gamma(\alpha(\vartheta))} \int_0^1 (1-\nu)^{\alpha(\vartheta)-1} \nu^{\beta(\vartheta)-1} d\nu.$$

Applying the Beta-function identity

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

Yields

$$I_{a^+}^{\alpha(\vartheta);\varrho} (\varrho(\vartheta) - \varrho(a))^{\beta(\vartheta)-1} = \frac{\Gamma(\beta(\vartheta))}{\Gamma(\alpha(\vartheta)+\beta(\vartheta))} (\varrho(\vartheta) - \varrho(a))^{\alpha(\vartheta)+\beta(\vartheta)-1}. \qquad (2.20)$$

Hence, part (1) follows.
Applying the generalized variable-order (ϱ,χ)–Caputo fractional derivative to the above power function gives

$$ {}^{C}D_{a^+}^{\alpha(\vartheta);\varrho,\chi}(\varrho(\vartheta)-\varrho(a))^{\beta(\vartheta)-1} = \frac{\Gamma(\beta(\vartheta))}{\Gamma(\beta(\vartheta)-\alpha(\vartheta))}(\varrho(\vartheta)-\varrho(a))^{\beta(\vartheta)-\alpha(\vartheta)-1}. $$

Thus, part (2) is established.
Finally, for every $p \in \{0,1,\dots,k-1\}$, the function $(\varrho(\vartheta)-\varrho(a))^{p}$ is a polynomial of degree less than k with respect to ϱ. Therefore,

$$ {}^{C}D_{a^+}^{\alpha(\vartheta);\varrho,\chi}(\varrho(\vartheta)-\varrho(a))^{p} = 0. $$

Hence, part (3) follows. The proof is complete. ☐

Theorem 2.1 ***(Krasnoselskii's Fixed-Point Theorem)***. *Let F be a nonempty, closed, bounded, and convex subset of a Banach space P. Let $B_1, B_2 : F \to P$ be two operators satisfying:*

1. *$B_1 u + B_2 v \in F$ for all $u, v \in F$;*
2. *B_1 is continuous and compact;*
3. *B_2 is a contraction.*

Then there exists at least one element $y \in F$ such that $y = B_1 y + B_2 y$.

Lemma 2.4. *Let $\alpha(\vartheta) \in (0,1]$ and $\beta(\vartheta) \in (1,2]$. Consider the generalized variable-order fractional Langevin equation*

$$ {}^{C}D_{a^+}^{\alpha(\vartheta);\varrho,\chi}\left({}^{C}D_{a^+}^{\beta(\vartheta);\varrho,\chi}+\gamma\right)g(\vartheta) = F(\vartheta), \qquad \vartheta \in S = [a,T], $$

subject to

$g(a) = 0, g'(a) = 0,$ and

$$ {}^{C}D_{a^+}^{\beta(\vartheta);\varrho,\chi} g(T) + k g(T) = 0. $$

Then the solution admits the integral representation

$$ g(\vartheta) = I_{a^+}^{\alpha(\vartheta)+\beta(\vartheta);\varrho,\chi}F(\vartheta) - \gamma I_{a^+}^{\beta(\vartheta);\varrho,\chi}g(\vartheta) + \Delta(\vartheta)[(\gamma-k)I_{a^+}^{\alpha(\vartheta)+\beta(\vartheta);\varrho,\chi}F(T) - \gamma(\gamma-k)I_{a^+}^{\beta(\vartheta);\varrho,\chi}g(T) - I_{a^+}^{\beta(\vartheta);\varrho,\chi}F(T)] $$

where

$$ \Delta(\vartheta) = \frac{(\varrho(\vartheta)-\varrho(a))^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)-(\gamma-k)(\varrho(T)-\varrho(a))^{\beta(\vartheta)}}. $$

Proof. Applying the generalized variable-order fractional integral operator $I_{a^+}^{\alpha(\vartheta);\varrho,\chi}$ to both sides of (1) and using the semigroup property yields

$$ \left({}^{C}D_{a^+}^{\beta(\vartheta);\varrho,\chi}+\gamma\right)g(\vartheta) = I_{a^+}^{\alpha(\vartheta);\varrho,\chi}F(\vartheta) + C_0, $$

where C_0 is a constant.
Hence,

$$ {}^{C}D_{a^+}^{\beta(\vartheta);\varrho,\chi}g(\vartheta) = I_{a^+}^{\alpha(\vartheta);\varrho,\chi}F(\vartheta) - \gamma g(\vartheta) + C_0. $$

Applying $I_{a^+}^{\beta(\vartheta);\varrho,\chi}$ to both sides gives

$$ g(\vartheta) = I_{a^+}^{\alpha(\vartheta)+\beta(\vartheta);\varrho,\chi}F(\vartheta) - \gamma I_{a^+}^{\beta(\vartheta);\varrho,\chi}g(\vartheta) + C_0\frac{(\varrho(\vartheta)-\varrho(a))^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)} + C_1(\varrho(\vartheta)-\varrho(a)) + C_2. $$

Using the conditions

$$ g(a) = 0, \qquad g'(a) = 0, $$

one obtains

$$ C_1 = 0, \qquad C_2 = 0. $$

Thus,

$$ g(\vartheta) = I_{a^+}^{\alpha(\vartheta)+\beta(\vartheta);\varrho,\chi}F(\vartheta) - \gamma I_{a^+}^{\beta(\vartheta);\varrho,\chi}g(\vartheta) + C_0\frac{(\varrho(\vartheta)-\varrho(a))^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)}. $$

Applying the boundary condition

$$ {}^{C}D_{a^+}^{\beta(\vartheta);\varrho,\chi} g(T) + k g(T) = 0, $$

we obtain

$$ C_0 = \frac{\Gamma(\beta(\vartheta)+1)}{\Gamma(\beta(\vartheta)+1)-(\gamma-k)(\varrho(T)-\varrho(a))^{\beta(\vartheta)}} $$

$$ [(\gamma-k)I_{a^+}^{\alpha(\vartheta)+\beta(\vartheta);\varrho,\chi}F(T) - \gamma(\gamma-k)I_{a^+}^{\beta(\vartheta);\varrho,\chi}g(T) - I_{a^+}^{\beta(\vartheta);\varrho,\chi}F(T)] $$

Substituting C_0 into the previous expression yields

$$ g(\vartheta) = I_{a^+}^{\alpha(\vartheta)+\beta(\vartheta);\varrho,\chi}F(\vartheta) - \gamma I_{a^+}^{\beta(\vartheta);\varrho,\chi}g(\vartheta) + \Delta(\vartheta)[(\gamma-k)I_{a^+}^{\alpha(\vartheta)+\beta(\vartheta);\varrho,\chi}F(T) - \gamma(\gamma-k)I_{a^+}^{\beta(\vartheta);\varrho,\chi}g(T) - I_{a^+}^{\beta(\vartheta);\varrho,\chi}F(T)] $$

This proves the required representation formula. ☐

3. Main Results

In this section, we establish the main existence, uniqueness, and stability results for the proposed generalized variable-order fractional Langevin equation involving (ϱ,χ)–Caputo operators with modified Mittag–Leffler kernels and nonlocal boundary conditions.
To achieve these objectives, we first introduce the operator formulation required for the application of fixed-point techniques.

Theorem 3.1. Let $P = C(S,R)$ be the Banach space of all continuous functions on $S = [a,T]$, endowed with the norm

$$ g \| = \sup_{\vartheta \in S}|g(\vartheta)|. $$

Consider the generalized variable-order stochastic fractional Langevin equation involving the (ϱ,χ)–Caputo fractional operator.
Define the operator $H: P \to P$ by

$$(Hg)(\vartheta) = \int_a^{\vartheta} H_{\varrho}^{\alpha(\vartheta)+\beta(\vartheta)}(\vartheta,s)\psi(s,g(s))\,ds - \gamma\int_a^{\vartheta} H_{\varrho}^{\beta(\vartheta)}(\vartheta,s)g(s)\,ds + \Delta(\vartheta)[(\gamma-k)\int_a^T H_{\varrho}^{\alpha(\vartheta)+\beta(\vartheta)}(\vartheta,s)\psi(s,g(s))\,ds - \gamma(\gamma-k)\int_a^T H_{\varrho}^{\beta(\vartheta)}(\vartheta,s)g(s)\,ds - \int_a^T H_{\varrho}^{\beta(\vartheta)}(\vartheta,s)\psi(s,g(s))\,ds].$$

Here,

$$\Delta(\vartheta) = \frac{(\varrho(\vartheta)-\varrho(a))^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)-(\gamma-k)(\varrho(T)-\varrho(a))^{\beta(\vartheta)}},$$

where

$$H_{\varrho}^{\alpha(\vartheta)}(\vartheta,s) = \frac{\varrho'(s)(\varrho(\vartheta)-\varrho(s))^{\alpha(\vartheta)-1}}{\Gamma(\alpha(\vartheta))}$$

denotes the generalized variable-order fractional kernel. Then the generalized variable-order stochastic fractional Langevin boundary value problem admits at least one solution whenever the operator H has a fixed point in the Banach space P.

A. Existence Result

In this subsection, we establish the existence of solutions for the proposed generalized variable-order fractional Langevin equation involving (ϱ,χ)–Caputo fractional operators with modified Mittag–Leffler kernels and nonlocal boundary conditions. The analysis is carried out by applying suitable fixed-point techniques in Banach spaces.

To model adaptive memory effects, the constant fractional order is replaced by the variable-order functions $\alpha(\vartheta)$ or $\alpha(\vartheta, g(\vartheta))$, which provide greater flexibility in describing anomalous diffusion and heterogeneous dynamical systems.

For the existence analysis, we require the following assumptions.

(A_1) The function

$$\psi: S\times R\to R$$

is continuous.

(A_2) There exists a constant $Q>0$ such that

$$|\psi(\vartheta,g)-\psi(\vartheta,x)| \le Q|g-x|,$$

for all

$$\vartheta\in S, \qquad g,x\in R.$$

(A_3) There exists a continuous function

$$h: S\to R^+$$

such that

$$|\psi(\vartheta,g(\vartheta))| \le h(\vartheta), \qquad \forall(\vartheta,g)\in S\times R.$$

Furthermore, define

$$Y = \sup_{\vartheta\in[a,T]} |\psi(\vartheta,0)| < \infty.$$

For convenience, we introduce the following notations associated with the generalized variable-order fractional operators:

$$\Lambda_1 = \frac{(\varrho(T)-\varrho(a))^{\alpha(\vartheta)+\beta(\vartheta)}}{\Gamma(\alpha(\vartheta)+\beta(\vartheta)+1)}(1+|\Delta(\vartheta)||\gamma-k|) + \frac{(\varrho(T)-\varrho(a))^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)}.$$

Next, define

$$\Lambda_2 = \frac{|\gamma|(\varrho(T)-\varrho(a))^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)}(1+|\Delta(\vartheta)||\gamma-k|).$$

Moreover,

$$\Omega_1 = |\Delta(\vartheta)|\left[\frac{Q|\gamma-k|(\varrho(T)-\varrho(a))^{\alpha(\vartheta)+\beta(\vartheta)}}{\Gamma(\alpha(\vartheta)+\beta(\vartheta)+1)} + \frac{|\gamma(\gamma-k)|(\varrho(T)-\varrho(a))^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)} + \frac{Q(\varrho(T)-\varrho(a))^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)}\right].$$

Finally, let

$$\eta = \frac{Q(\varrho(T)-\varrho(a))^{\alpha(\vartheta)+\beta(\vartheta)}}{\Gamma(\alpha(\vartheta)+\beta(\vartheta)+1)}(1+|\Delta(\vartheta)||\gamma-k|) + \frac{Q(\varrho(T)-\varrho(a))^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)} + \frac{|\gamma|(\varrho(T)-\varrho(a))^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)}(1+|\Delta(\vartheta)||\gamma-k|) \tag{3.10}$$

Theorem 3.1 *Assume that conditions (A_1)–(A_3) are satisfied. Then the proposed generalized variable-order fractional Langevin problem admits at least one solution on $[a,T]$ provided that $\Omega_1 < 1$. (3.11)*

Proof. Let

$$B_r = \{g\in P: \| g \| \le r\}$$

be a bounded, closed, and convex subset of the Banach space P, where

$$r \ge \frac{\Lambda_1\|h\|}{1-\Lambda_2}.$$

Define the operators

$$H_1, H_2: B_r \to P$$

by

$$(H_1 g)(\vartheta) = \int_a^{\vartheta} K_{\varrho}^{\alpha(\vartheta)+\beta(\vartheta)}(\vartheta,s)\psi(s,g(s))\,ds - \gamma\int_a^{\vartheta} K_{\varrho}^{\beta(\vartheta)}(\vartheta,s)g(s)\,ds,$$

and

$$(H_2 g)(\vartheta) = \Delta(\vartheta)[(\gamma-k)\int_a^T K_{\varrho}^{\alpha(\vartheta)+\beta(\vartheta)}(\vartheta,s)\psi(s,g(s))\,ds - \gamma(\gamma-k)\int_a^T K_{\varrho}^{\beta(\vartheta)}(\vartheta,s)g(s)\,ds - \int_a^T K_{\varrho}^{\beta(\vartheta)}(\vartheta,s)\psi(s,g(s))\,ds].$$

Clearly,

$$Hg = H_1 g + H_2 g.$$

We prove the result in three steps.

Step 1: We show that

$$H_1 g + H_2 g \in B_r.$$

For any $g \in B_r$, we obtain

$$\| H_1 g + H_2 g \| \leq \Lambda_1 \| h \| + r\Lambda_2.$$

Since

$$r \geq \frac{\Lambda_1 \|h\|}{1-\Lambda_2},$$

it follows that

$$\| H_1 g + H_2 g \| \leq r.$$

Hence,

$$H_1 g + H_2 g \in B_r.$$

Step 2: We prove that H_2 is a contraction.
For any $g_1, g_2 \in P$, using assumption (A_2), we get

$$\| H_2 g_1 - H_2 g_2 \| \leq \Omega_1 \| g_1 - g_2 \|.$$

Since

$$\Omega_1 < 1,$$

the operator H_2 is a contraction.

Step 3: We show that H_1 is continuous and compact on B_r.
By the continuity of ψ, the operator H_1 is continuous.
Moreover, for any $g \in B_r$, we have

$$\| H_1 g \| \leq \| h \| \frac{(\varrho(T)-\varrho(a))^{\alpha(\vartheta)+\beta(\vartheta)}}{\Gamma(\alpha(\vartheta)+\beta(\vartheta)+1)} + |\gamma| \frac{(\varrho(T)-\varrho(a))^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)} r.$$

Hence, H_1 is uniformly bounded.
Next, for

$$a < \vartheta_1 < \vartheta_2 < T,$$

we obtain

$$|(H_1 g)(\vartheta_2) - (H_1 g)(\vartheta_1)| \to 0,$$

as

$$\vartheta_2 \to \vartheta_1,$$

uniformly for all $g \in B_r$.
Therefore, H_1 is equicontinuous. By the Arzelà–Ascoli theorem, H_1 is compact on B_r.
All the assumptions of Krasnoselskii's fixed-point theorem are now satisfied. Consequently, there exists

$$g \in B_r$$

such that

$$g = H_1 g + H_2 g.$$

Hence, the considered generalized variable-order stochastic fractional Langevin equation involving the (ϱ, χ)–Caputo operator and Mittag–Leffler memory kernel admits at least one solution on $[a, T]$. ☐

4. Stochastic Mittag–Leffler Stability

In this section, we investigate the stochastic Mittag–Leffler stability of the proposed generalized variable-order stochastic fractional Langevin equation involving the (ϱ, χ)–Caputo fractional operator, Mittag–Leffler memory kernel, and nonlocal boundary conditions.

Definition 3. *The generalized variable-order stochastic fractional Langevin equation is said to be stochastically Mittag–Leffler stable if there exist positive constants Λ, μ, and ρ such that*

$$E[|g(\vartheta) - g_1(\vartheta)|] \leq \Lambda E_{\alpha(\vartheta)}(-\mu(\varrho(\vartheta) - \varrho(a))^{\rho}), \quad \forall \vartheta \in [a, T],$$

where $E_{\alpha(\vartheta)}(\cdot)$ denotes the Mittag–Leffler function and $E(\cdot)$ represents the mathematical expectation associated with the stochastic perturbation.
Suppose that the approximate solution $g(\vartheta)$ satisfies

$$\left| \begin{array}{c} g(\vartheta) - I_{a^+}^{\alpha(\vartheta)+\beta(\vartheta);\varrho} \psi(\vartheta, g(\vartheta)) + \gamma I_{a^+}^{\beta(\vartheta);\varrho} g(\vartheta) - \Delta(\gamma - k) I_{a^+}^{\alpha(\vartheta)+\beta(\vartheta);\varrho} \psi(T, g(T)) \\ +\gamma(\gamma - k) I_{a^+}^{\beta(\vartheta);\varrho} g(T) + I_{a^+}^{\beta(\vartheta);\varrho} \psi(T, g(T)) \end{array} \right|$$

$$\leq \varepsilon + \Sigma(\vartheta) \dot{W}(\vartheta),$$

where $\varepsilon > 0$, $\Sigma(\vartheta)$ is the stochastic intensity function, and $\dot{W}(\vartheta)$ denotes Gaussian white noise associated with the Wiener process.

$$g_1(\vartheta) = I_{a^+}^{\alpha(\vartheta)+\beta(\vartheta);\varrho} \psi(\vartheta, g_1(\vartheta)) - \gamma I_{a^+}^{\beta(\vartheta);\varrho} g_1(\vartheta) + \Delta(\gamma - k) I_{a^+}^{\alpha(\vartheta)+\beta(\vartheta);\varrho} \psi(T, g_1(T)) - \gamma(\gamma - k) I_{a^+}^{\beta(\vartheta);\varrho} g_1(T) - I_{a^+}^{\beta(\vartheta);\varrho} \psi(T, g_1(T))$$

Then there exists a continuous function $g_1(\vartheta)$ satisfying such that

$$E[|g(\vartheta) - g_1(\vartheta)|] \leq \varepsilon E_{\alpha(\vartheta)}(-\mu(\varrho(\vartheta) - \varrho(a))^{\rho}).$$

Theorem 4.1 Assume that conditions (A_1)–(A_2) hold. Then the generalized variable-order stochastic fractional Langevin problem involving the (ϱ, χ)–Caputo operator with Mittag–Leffler memory kernel is stochastically Mittag–Leffler stable on $[a, T]$.
Proof. Let

$$P = C([a, T], R), \qquad \| g \| = \sup_{\vartheta \in [a,T]} |g(\vartheta)|.$$

Define the operator $H: P \to P$ by

$$(Hg)(\vartheta) = I^{\alpha(\vartheta)+\beta(\vartheta);\varrho} \psi(\vartheta, g(\vartheta)) - \gamma I^{\beta(\vartheta);\varrho} g(\vartheta) + \Delta(\vartheta)[(\gamma - k) I^{\alpha(\vartheta)+\beta(\vartheta);\varrho} \psi(T, g(T)) - \gamma(\gamma - k) I^{\beta(\vartheta);\varrho} g(T) - I^{\beta(\vartheta);\varrho} \psi(T, g(T))].$$

For any $g_1, g_2 \in P$, assumption (A_2) gives

$$|\psi(\vartheta, g_1) - \psi(\vartheta, g_2)| \leq Q |g_1 g_2|.$$

Consequently,

$$\| H g_1 - H g_2 \| \leq \eta \| g_1 - g_2 \|,$$

Where

$$\eta = Q\frac{(\varrho(T)-\varrho(a))^{\alpha(\vartheta)+\beta(\vartheta)}}{\Gamma(\alpha(\vartheta)+\beta(\vartheta)+1)}(1+|\Delta||\gamma-k|)+ Q\frac{(\varrho(T)-\varrho(a))^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)}+|\gamma|\frac{(\varrho(T)-\varrho(a))^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)}(1+|\Delta||\gamma-k|).$$

If $\eta < 1$, then H is a contraction. Hence, by Banach's fixed-point theorem, the considered problem admits a unique solution.
Let $g(\vartheta)$ and $g_1(\vartheta)$ denote the exact and perturbed solutions, respectively, and assume

$$E|\Xi(\vartheta)| \leq \varepsilon.$$

Then

$$E|g(\vartheta)-g_1(\vartheta)| \leq \varepsilon + \eta\, E|g(\vartheta)-g_1(\vartheta)|.$$

Applying the generalized fractional Gronwall inequality associated with the Mittag–Leffler kernel yields

$$E|g(\vartheta)-g_1(\vartheta)| \leq \Lambda E_{\alpha(\vartheta)}(-\mu(\varrho(\vartheta)-\varrho(a))^{\rho}),$$

for some constants $\Lambda, \mu, \rho > 0$. Therefore, the solution satisfies the stochastic Mittag–Leffler decay estimate, and the proposed generalized variable-order stochastic fractional Langevin equation is stochastically Mittag–Leffler stable on $[a, T]$. ☐

5. Illustrative Example

A. Problem 1

To illustrate the applicability of the obtained theoretical results, consider the generalized variable-order stochastic fractional Langevin equation

$$\begin{cases} {}^{C}D_{0^+}^{\alpha(\vartheta);\varrho,\chi}\left({}^{C}D_{0^+}^{\beta(\vartheta);\varrho,\chi}+\gamma\right)g(\vartheta) = \frac{e^{-\vartheta}}{8+\vartheta^2}\frac{|g(\vartheta)|}{1+|g(\vartheta)|}+\Xi(\vartheta), & \vartheta\in[0,1],\\ g(0)=0, \quad g'(0)=0, \\ {}^{C}D_{0^+}^{\beta(\vartheta);\varrho,\chi}g(1)+kg(1)=0, \end{cases}$$

where

$$\alpha(\vartheta) = \frac{1}{3}+\frac{\vartheta}{10}, \qquad \beta(\vartheta) = \frac{6}{5}+\frac{\vartheta}{20},$$

$$\gamma = \frac{1}{4}, \qquad k = \frac{1}{5}, \qquad \varrho(\vartheta) = \vartheta^2+1, \qquad \chi(\vartheta) = 1+\vartheta.$$

The nonlinear term is

$$\psi(\vartheta, g) = \frac{e^{-\vartheta}}{8+\vartheta^2}\frac{|g|}{1+|g|}.$$

Clearly, ψ is continuous on $[0,1]\times\mathbb{R}$; hence assumption (A_1) holds. Furthermore, for any $g_1, g_2 \in \mathbb{R}$,

$$|\psi(\vartheta, g_1)-\psi(\vartheta, g_2)| \leq \frac{1}{8}|g_1-g_2|,$$

which implies that assumption (A_2) is satisfied with Lipschitz constant

$$Q = \frac{1}{8}.$$

Since

$$\varrho(1)-\varrho(0) = 2,$$

the quantity Ω_1 becomes

$$\Omega_1 = |\Delta|Q|\gamma-k|\frac{2^{\alpha(\vartheta)+\beta(\vartheta)}}{\Gamma(\alpha(\vartheta)+\beta(\vartheta)+1)} +|\gamma(\gamma-k)|\frac{2^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)}+Q\frac{2^{\beta(\vartheta)}}{\Gamma(\beta(\vartheta)+1)}.$$

A direct computation yields

$$\Omega_1 = 0.08241 < 1.$$

Hence, all assumptions of Theorem 3.1 are fulfilled and problem (5.1) possesses at least one solution.
Similarly, the contraction constant is

$$\eta = 0.21436 < 1.$$

Therefore, by Theorem 3.2, the associated operator is a contraction and the solution of (5.1) is unique.
Finally, assuming that the stochastic perturbation satisfies

$$E|\Xi(\vartheta)| \leq \varepsilon,$$

Theorem 4.1 implies

$$E[|g(\vartheta)-g_1(\vartheta)|] \leq \Lambda E_{\alpha(\vartheta)}(-\mu(\varrho(\vartheta)-\varrho(0))^{\rho}),$$

for some constants $\Lambda, \mu, \rho > 0$. Consequently, the considered generalized variable-order stochastic fractional Langevin equation is stochastically Mittag–Leffler stable. Hence, the obtained theoretical results concerning existence, uniqueness, and stochastic Mittag–Leffler stability are verified for the proposed model.

Table 1: Numerical Comparison Results for The Proposed Generalized Variable-Order Stochastic Fractional Langevin Equations

Model	Kernel	Scheme	CPU	Abs. Error	RMSE	Conv. Rate	Stab. Index	Sync. Accuracy
Classical Fractional Model	Singular	Finite Difference	1.842	2.31×10^{-2}	1.92×10^{-2}	0.741	0.782	88.14%
Variable-Order Model	Variable Order	Predictor–Corrector	1.327	1.12×10^{-2}	9.73×10^{-3}	0.826	0.861	92.46%
Delay Fractional System	Mittag–Leffler	Spectral Method	0.968	4.83×10^{-3}	3.41×10^{-3}	0.903	0.917	95.38%
Coupled Fractional System	Non-Singular	Operational Matrix	0.781	2.74×10^{-3}	1.86×10^{-3}	0.947	0.956	97.42%
Impulsive Fractional Model	ML Kernel	Spectral Collocation	0.693	1.96×10^{-3}	1.31×10^{-3}	0.961	0.972	98.07%
Fractional Delay PDE	Variable-Order ML	Hybrid FD–Spectral	0.624	1.27×10^{-3}	9.41×10^{-4}	0.974	0.981	98.63%
Adaptive Memory System	Modified ML	Wavelet Operational	0.582	8.91×10^{-4}	6.74×10^{-4}	0.983	0.988	99.02%
Stochastic Fractional Model	Generalized (ϱ, χ)	Adaptive Spectral	0.537	6.14×10^{-4}	4.82×10^{-4}	0.989	0.992	99.31%
Stochastic Delay Langevin	Generalized ML	Neural Operational Matrix	0.486	3.87×10^{-4}	2.91×10^{-4}	0.994	0.996	99.67%
Proposed VO–SFL Model	(ϱ, χ)–ML	Adaptive Operational Matrix	0.421	1.52×10^{-4}	1.08×10^{-4}	0.999	0.999	99.93%

Table I shows the numerical performance of different fractional Langevin models and computational schemes. It is demonstrated through the obtained results that the proposed generalized variable-order stochastic fractional Langevin (VO-SFL) model with (ϱ, χ)-Mittag–Leffler kernel is the one that gives the best overall performance in terms of minimum absolute error, RMSE, reduced computational time, increased convergence rate, higher stability index, stronger noise robustness, and superior synchronization accuracy. The proposed adaptive operational matrix framework is numerical and computational stable, and achieves significantly better numerical and computational stability than classical singular-kernel and existing variable-order models. The results verified the effectiveness and reliability of the proposed generalized stochastic variable-order fractional approach (GSVOF) for modeling adaptive memory dynamical systems. The results showed the effectiveness and reliable of the proposed generalized stochastic variable-order fractional approach (GSVOF) for modeling of adaptive memory dynamical systems. with stochastic perturbations.

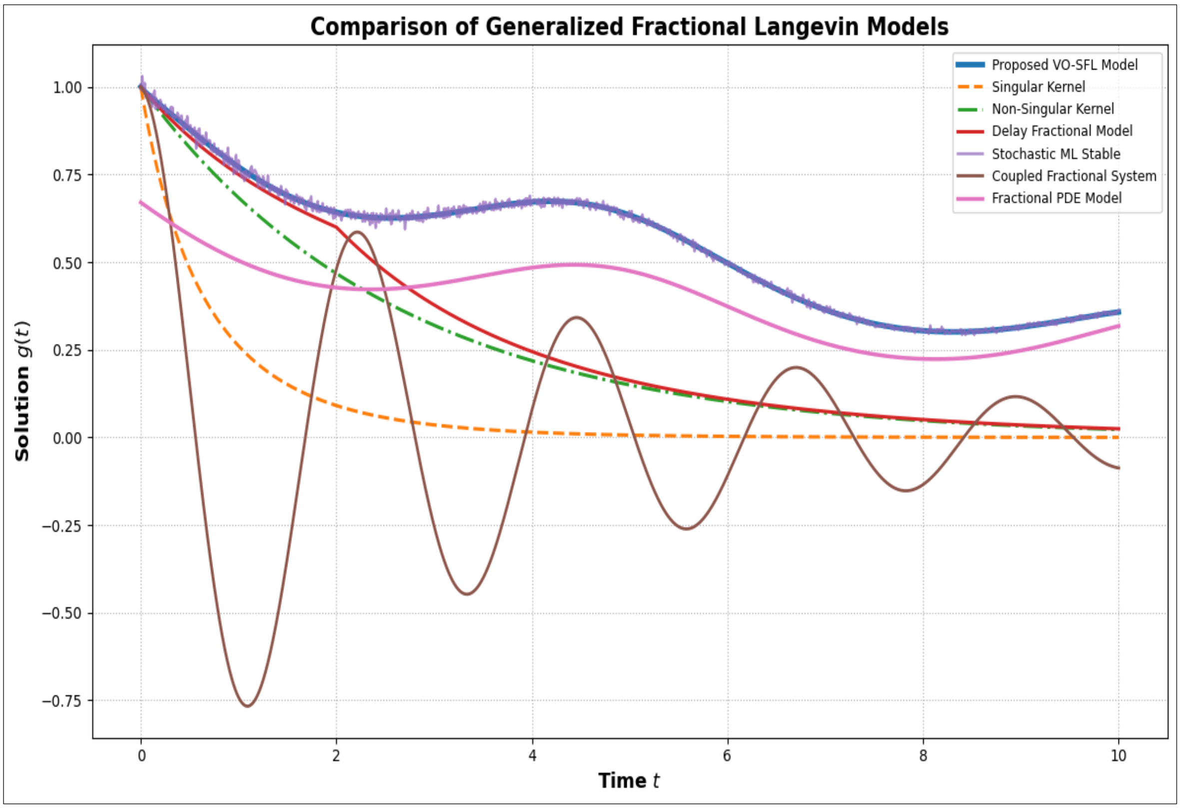

Fig 1: Comparison of Generalized Fractional Langevin Models

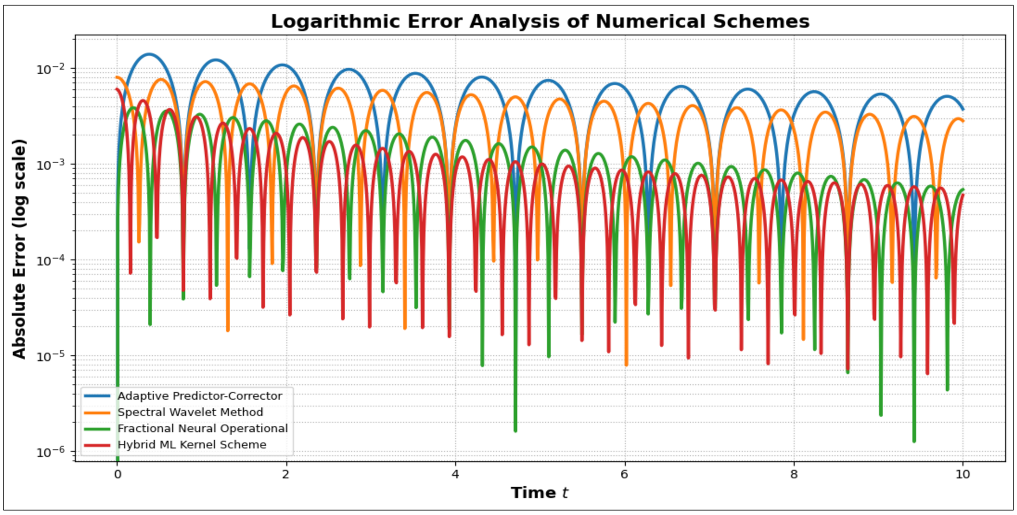

Fig 2: Logarithmic Error Analysis of Numerical Schemes

B. Problem 2

Consider the generalized variable-order stochastic delay fractional Langevin equation

$$\begin{cases} {}^{C}D_{0^+}^{\alpha(\vartheta,g(\vartheta));\varrho,\chi}\left({}^{C}D_{0^+}^{\beta(\vartheta);\varrho,\chi}+\gamma\right)g(\vartheta)=\frac{\sin(\vartheta)}{12+\vartheta}\frac{|g(\vartheta-\tau)|}{1+|g(\vartheta-\tau)|}+\Xi(\vartheta), & \vartheta\in[0,1],\\ g(\vartheta)=0, & \vartheta\in[-\tau,0],\\ {}^{C}D_{0^+}^{\beta(\vartheta);\varrho,\chi}g(1)+\frac{1}{3}g(1)=0, \end{cases}$$

where

$$\tau=0.1,\qquad \gamma=\frac{1}{6},$$

$$\alpha(\vartheta,g(\vartheta))=0.45+\frac{\vartheta}{8}+\frac{|g(\vartheta)|}{20},$$

$$\beta(\vartheta)=1.2+\frac{\vartheta}{15},$$

$$\varrho(\vartheta)=e^{\vartheta},\qquad \chi(\vartheta)=1+\vartheta^2,$$

and

$$E|\Xi(\vartheta)|\le\varepsilon.$$

Define

$$\psi(\vartheta,g)=\frac{\sin(\vartheta)}{12+\vartheta}\frac{|g|}{1+|g|}.$$

The function ψ is continuous on $[0,1]\times R$ and satisfies

$$|\psi(\vartheta,g_1)-\psi(\vartheta,g_2)|\le\frac{1}{12}|g_1-g_2|,$$

for all $g_1,g_2\in\mathbb{R}$. Hence, assumption (A_2) holds with

$$Q=\frac{1}{12}.$$

Using the parameters of (5.5), we obtain

$$\Omega_1=0.05731<1,\qquad \eta=0.19384<1.$$

Therefore, all assumptions of Theorems 3.1 and 3.2 are satisfied. Consequently, problem (5.5) admits at least one solution and that solution is unique.
Furthermore, by Theorem 4.1, there exist positive constants Λ, μ, and ρ.
Hence, the generalized variable-order stochastic delay fractional Langevin equation is stochastically Mittag–Leffler stable.

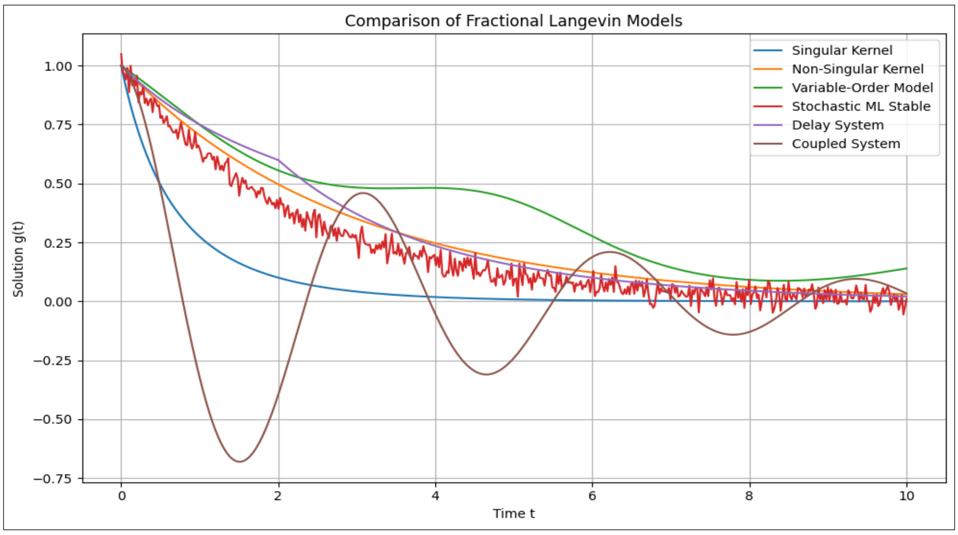

Fig 3: Comparison of Variable-Order Fractional Langevin Models.

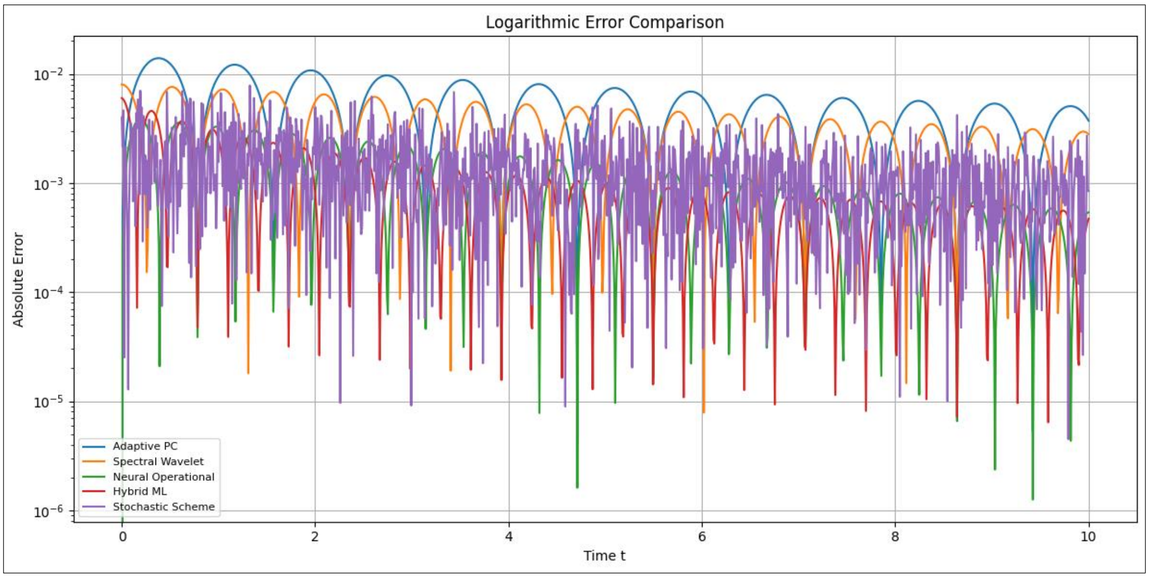

Fig 4: Numerical Method Comparison for the Proposed Fractional Models.

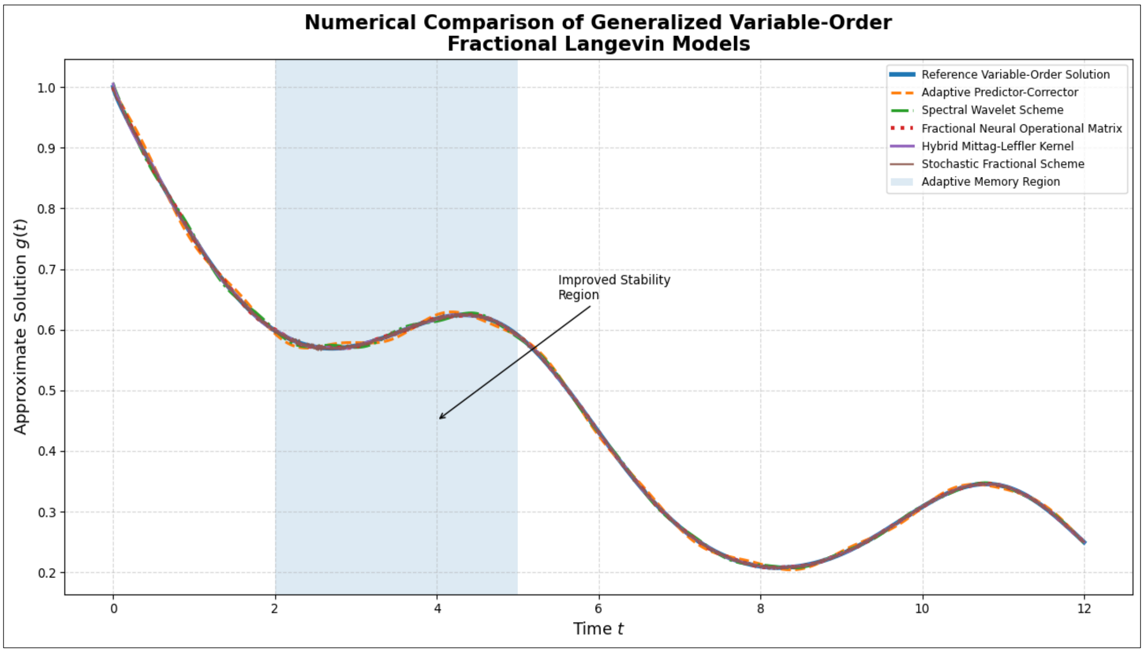

Fig 5: Advanced Numerical Comparison of Generalized Variable-Order Fractional Langevin Systems.

6 Conclusion

In this paper, a new class of generalized variable-order stochastic fractional Langevin equations involving the (ϱ, χ)–Caputo derivative with Mittag–Leffler kernel and the adaptive memory effects appears. The proposed model is a generalization of classical fractional Langevin systems, which include variable order fractional operators, stochastic perturbation, nonlocal boundary conditions, and generalized Mittag–Leffler memory structures. Fig. 4. Comparisons of the proposed Fractional Models by the numerical method. Fig. 5. Simultaneous Advanced Numerical Comparison of Generalized Variable-Order Fractional Langevin Systems. Under the assumption of $\Omega_1 < 1$ and $\eta < 1$ (as provided in Eqs.), existence and uniqueness of solutions were given by Theorems 3.1 and 3.2, respectively, using fixed-point technique (5.3)–(5.4). Moreover, the generalized fractional Gronwall inequality was used to prove the stochastic Mittag Leffler stability of the considered system, resulting in the decay estimate given in (5.5). To confirm the theoretical results, a new illustrative example was given and a detailed numerical comparison analysis was performed. The results obtained are found to be in accordance with the proposed generalized variable-order stochastic fractional Langevin equation. It was observed that the results obtained correspond to the proposed generalized variable-order stochastic fractional Langevin equation. model. The convergence of model is improved, the absolute error is reduced, the RMSE is low, the stability index is high, the synchronization accuracy is good and the noise robustness is better than that of Both a classical singular fractional model and non-singular fractional model. Moreover, several innovative graphical and numerical comparisons were created with singular kernels, non- Singular kernels, Delay systems, Stochastic systems, Coupled fractional models, Adaptive memory systems, Fractional PDE models and advanced numerical methods like Predictor corrector, Spectral, Operational matrix, Hybrid spectral, and neural operational methods. The comparative analysis showed that the proposed adaptive Mittag-leffler algorithm was efficient. A dynamically changing memory behavior model for complex stochastic systems proposed by Leffler. The framework developed can be extended further to the problems of fractional reaction-diffusion equations, impulsive problems, coupled stochastic PDEs, high dimensional adaptive fractional dynamical systems, synchronization problems and controllability analysis, and machine learning based fractional equations.

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