



Dynamic Production Inventory Systems with Deterioration: Nonlinear Modelling and Game-Theoretic Extensions

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Abstract

This study develops and analyses a nonlinear production inventory model for non-instantaneous deteriorating items under time-dependent production and demand rates. The model is further extended into a differential game setting to capture strategic interactions between production and demand agents. Analytical solutions are derived for the deterministic model, while the game-theoretic system is solved numerically using a forward-backward sweep method. Cost functionals and optimal cycle time are studied. The results demonstrate that incorporating dynamic controls and strategic interactions significantly improves modelling realism and decision-making efficiency.

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1. Introduction

Inventory systems for deteriorating items have received considerable attention due to their applications in food storage, pharmaceuticals, and supply chains (Ghare & Schrader, 1963; Goyal & Giri, 2001) ^[16, 18]. Classical models assume constant production and demand, which is unrealistic in dynamic markets.

Recent advances incorporate time-dependent rates and nonlinear effects, yet most neglect strategic interactions among decision-makers. This work bridges this gap by integrating nonlinear dynamics with game theory. The classical production inventory model for non-instantaneous deteriorating items assumes constant production rate, constant demand, and constant deterioration. In the referenced work of Garba *et al.* (2025) ^[15], the governing equations are:

$$\begin{cases} \frac{dI_1}{dt} + \mu I_1 = a - d, & 0 \leq t \leq T_1 \\ \frac{dI_2}{dt} + \nu I_2 = -d, & T_1 \leq t \leq T \end{cases} \quad (1.1)$$

where production a and demand d rates are constant and deterioration occurs only after production stops. The other parameters including μ as average production rate, ν represents the deterioration rate, T_1 the stopping time of production and the beginning of deterioration, T the cycle length time, A_0 the start-up cost, c the unit cost of production, $I_1(t)$ inventory level during production, $I_2(t)$ inventory level after production stops, T_c total variable cost per unit time, and h the holding cost. (Note that $\mu = 0$ in the work by Garba *et al.* (2025)) ^[15]. However, real-world inventory systems often exhibit:

- Time-dependent production rates (machine variability, workforce shifts, etc.);
- Time-dependent demand (seasonality, market fluctuations); and
- Proportional inventory growth/decay dynamics.

1.2 Definition of Key Terms

Differential Game: represents a dynamic game where players make decisions continuously over time, and the system's evolution is described by differential equations in differential form such as $dl = Mdx + Ndy$.

Nash Equilibrium: In a game, a set of strategies where no player can unilaterally improve their outcome by changing their own strategy, assuming the other players' strategies remain fixed.

Cost Functional (Payoff Functional): A mathematical expression that assigns a numerical value (cost or profit) to a player's strategy over a period. In our settings, these are defined as integrals of a running payoff.

Hamiltonian: This is a central function in optimal control theory that combines the running payoff with the dynamics of the state variables via costate (adjoint) variables.

Costate Variables (Adjoint Variables): This is a dynamic multiplier that represents the marginal value (or "shadow price") of the state variable at any point in time.

2. Related & Previous Studies

Inventory models for deteriorating items have evolved significantly over the past few decades, as they can be seen in this work, incorporating increasingly realistic assumptions such as time-dependent demand, imperfect production, and strategic decision-making.

Some of the early advancements in nonlinear inventory systems include the work of Sarkar (2012)^[38], who developed production-inventory models under reliability considerations and imperfect processes, highlighting the importance of system efficiency in deteriorating environments. Similarly, Khouja (2010)^[27] examined lot-sizing problems under variable demand and deterioration, providing foundational insights into time-dependent inventory behaviour. Game-theoretic approaches have been extensively studied in economic and operational systems. The seminal work of Dockner *et al.* (2000)^[14] introduced differential games in dynamic optimization, while Jørgensen and Zaccour (2004)^[23] provided a comprehensive treatment of supply chain coordination using game-theoretic settings. These works laid the great foundation for integrating strategic interactions into inventory models.

By incorporating financial and operational constraints, Alfares (2007)^[3] analyzed inventory models with stock-dependent demand, while Teng (2006)^[42] introduced economic order quantity models under permissible delay in payments, improving the financial realism of inventory systems.

Recent studies have focused on nonlinear and time-dependent extensions. Yang (2015)^[45] and Wu (2019)^[44] investigated inventory systems with dynamic demand and deterioration rates, demonstrating improved adaptability to market fluctuations. Taleizadeh *et al.* (2018)^[19] further extended these models to include sustainability considerations.

In the context of uncertainty and stochastic modelling, Paul *et al.* (2021)^[26] and Boute *et al.* (2014)^[47] explored probabilistic demand and supply disruptions, emphasizing robustness in inventory control. Similarly, Li *et al.* (2021) and Kutunoglu & Lohiya (2008)^[8] examined multi-echelon and integrated supply chain systems.

Optimization and computational approaches have also gained attention. Roy *et al.* (2019)^[37] and Bakker *et al.* (2012)^[5] proposed numerical techniques for solving nonlinear inventory models, while Fatemi *et al.* (2020)^[28] and

Govindan *et al.* (2017)^[46] applied advanced optimization methods to large-scale systems.

Furthermore, Priore *et al.* (2019)^[35], Çaylı and Oralhan, (2024)^[1], and Sadeghi *et al.* (2016)^[32] contributed to the development of hybrid inventory models incorporating machine learning and adaptive control strategies.

Recent reviews by Asghari *et al.* (2026)^[4] and Becerra *et al.* (2022)^[13] provide comprehensive overviews of sustainable and deteriorating inventory models, highlighting the growing importance of environmental and strategic factors.

Despite these advancements, there remains a gap in combining nonlinear time-dependent inventory dynamics with differential game theory. This study addresses this gap by integrating a two-phase deteriorating inventory model with a Nash equilibrium implementation.

The study of deteriorating inventory systems began with the seminal work of Ghare and Schrader (1963)^[16], who introduced deterioration (exponential decay type) into EOQ inventory dynamics. Subsequent developments incorporated more realistic demand structures, including time-dependent demand and deterioration rates, as systematically reviewed by Goyal and Giri (2001)^[18], who classified inventory models based on demand variability and deterioration characteristics. The case in which deterioration rate is governed by Weibull distribution function was studied by Chakrabarty *et al.* (1998)^[24]. The non-instantaneous deterioration rate case was studied in 2012 by Sarkar. Earlier, the incorporation of differential game applications for such models was introduced by Dockner *et al.* (2000)^[14]. The investigation of supply chain Nash equilibria was analysed by Jørgensen and Zaccour (2004)^[23]. More recent surveys highlight the rapid expansion of the field, with over 200 contributions addressing stochastic demand, supply chains, and multi-echelon systems (Bakker *et al.*, 2012)^[11].

Modern research has extended these models to incorporate sustainability, adaptive systems, and data-driven approaches, reflecting emerging real-world complexities in an inventory management (Asghari *et al.*, 2026)^[4].

However, few studies combine nonlinear inventory evolution with dynamic game strategies. This article aims to develop and analyse nonlinear inventory models with strategic decision-making through the following objectives:

- Formulate a linear and nonlinear two-phase inventory model;
- Derive analytical solutions;
- Extend the model into a Nash differential game;
- Develop numerical solution schemes;
- Analyse cost and optimal cycle time.

3. Methods

3.1. Model Formulation

Consider the classical Production model (1.1) and its limitations. To address the highlighted limitations, we propose the extended model:

$$\frac{dl_1}{dt} - [\alpha(t) - d(t)]I_1 = \gamma(t), \quad 0 \leq t \leq T_1 \quad (3.1)$$

$$\frac{dl_2}{dt} + v I_2 = -d(t), \quad T_1 \leq t \leq T \quad (3.2)$$

with the boundary conditions:

$$I_1(0) = I_0, \quad I_2(T_1) = I_d \quad (3.3)$$

The parameter $\alpha(t)$ is the time-dependent production rate, $d(t)$ the instantaneous demand rate, $\mu(t) = -[\alpha(t) - d(t)]$ represents the net production rate, $\gamma(t)$ is an external replenishment that enhances production rate, and ν is the deterioration rate. Equation (3.1) introduces a multiplicative (proportional) growth structure, where:

- $\alpha(t)I_1$ represents the production contribution scales with current inventory level, capturing efficiency scaling in production systems, batch-dependent processing, and feedback-controlled production systems.
- $d(t)I_1$: is the demand reduces inventory proportionally, representing demand intensity relative to stock availability and scenarios where higher stock visibility increases sales.

Unlike the classical linear model $a - d$, this formulation captures nonlinear growth dynamics, allows time-varying production and demand, and reflects inventory-dependent consumption behaviour.

For the second equation (3.2), we have:

- νI_2 representing the exponential deterioration common in perishable goods, pharmaceuticals, and food storage systems.
- $d(t)$: Time-dependent demand reflecting seasonal demand and market fluctuations.

To match the two boundaries, we impose the condition that $I_2(T_1) = I_d$. This condition ensures continuity between production and depletion phases.

This extended model improves the original in the following ways:

1. **Generalization of Constants:** production rate $a \rightarrow \alpha(t)$ and demand rate $d \rightarrow d(t)$ for $t \in \mathbb{R}$.
2. **Nonlinear Dynamics:** Captures realistic growth and decay patterns, allowing richer analytical and numerical behaviour.
3. **Better Representation of Real Systems:** Industrial production is rarely constant, demand is rarely constant, and deterioration often follows exponential decay.
4. **Mathematical Flexibility:** The model still accommodates the special cases $\alpha(t) = a$, reducing to the classical model.

The proposed extension transforms the classical production inventory model into a more realistic and flexible setting capable of capturing nonlinear, time-dependent, and dynamic inventory behaviours. It bridges the gap between theoretical assumptions and practical industrial systems. The model can be further analysed based on their phases. *Phase I* shows growth dominated by production; while *Phase II* highlights decay dominated by deterioration. Altogether, the stability of the system depends on the sign of $\alpha(t) - d(t)$.

Table 1: List of symbols and their descriptions.

Symbols	Description/meaning
I_0	Initial Inventory;
I_d	Initial degraded Inventory;
I_1	Inventory level of a product stock or in a supply chain/amount of raw materials at time t ;
I_2	Inventory level at the beginning of deterioration for $t \geq T_1$;
$\alpha(t)$	Instantaneous production rate;
$d(t)$	Instantaneous demand rate;
$\mu(t)$	Amplification/proportional depletion of current inventory;
$\gamma(t)$	External demand incentive (subsidy, marketing stimulation);
c_p	Production cost coefficient;
c_d	Demand/sales cost coefficient;
h	Holding cost coefficient;
p	Unit profit parameter;
J_M	Manufacturer cost;
J_R	Retailer's profit.

2.2. Analytical Solutions: Deterministic Two-Phase Model.

The two-phase deterministic model admits analytical solutions:

Phase I: The model equation (3.1), via the integrating factor method, admits the solution

$$I_1(t) = \left[I_0 + \int_0^t \gamma(s) e^{-\int_0^s [\alpha(s) - d(s)] ds} ds \right] e^{\int_0^t [\alpha(s) - d(s)] ds} \quad (3.4)$$

Phase II: Similarly, the solution to the second model equation (2.2) reads

$$I_2(t) = \left[I_d - \int_0^t d(s) e^{\int_{T_1}^s \nu(s) ds} ds \right] e^{-\int_{T_1}^t \nu(s) ds} \quad (3.5)$$

To capture all possible variations of the involved parameters, we take $\gamma(t) = 2 + 0.5 \sin(2t)$, $\nu(t) = 0.4 + 0.15 \sin(t)$, $\alpha(t) = 0.6 + 0.2 \sin(t)$, and $d(t) = 0.3 + 0.1 \cos(t)$. The solutions for these parameters are illustrated in Figures 1–3. Note that in Figure 1, the boundaries are not matched.

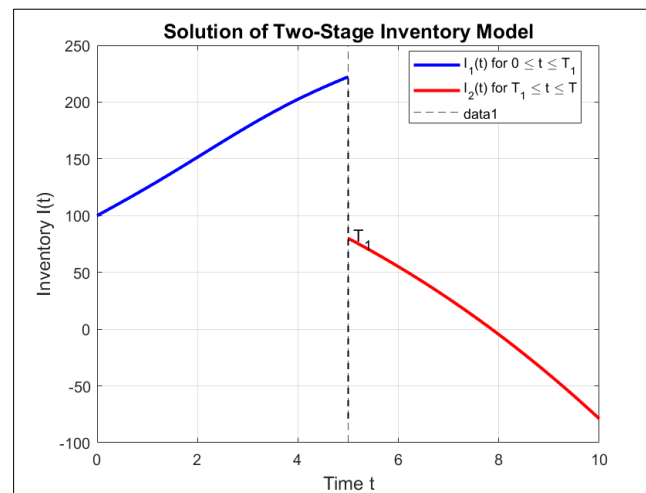


Fig 1: Two-stage solution for $\gamma(t) = 2 + 0.5 \sin(2t)$, $v(t) = 0.4 + 0.15 \sin(t)$, $\alpha(t) = 0.6 + 0.2 \sin(t)$, $d(t) = 0.3 + 0.1 \cos(t)$, and $0 \leq t \leq 10$.

The choice of the functions $\gamma(t)$, $\alpha(t)$ and $d(t)$ is meant to capture the increase and decrease of the parameters as the $\sin$ and $\cos$ functions possess these properties.

The results of these Figures 1-3 are the solutions of the equations (3.4) – (3.5).

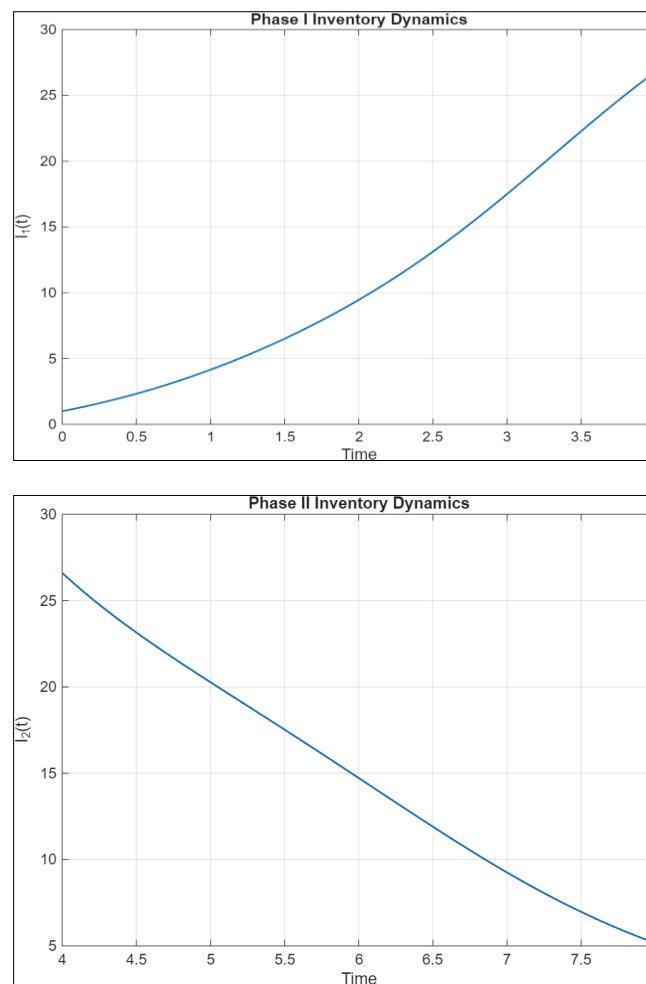


Fig 2: Inventory dynamics for $I_1(t)$ and $I_2(t)$ with $I_0 = 1$ and $I_d = 26.6$.

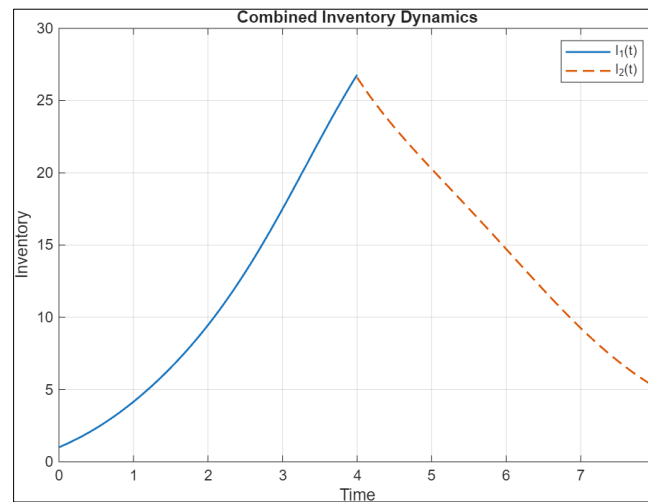


Fig 3: Combined inventory dynamics for both phases, $I_0 = 1$, and $I_d = 26.6$.

3.3. Cost Functional and Optimal Cycle Time

The formulation of cost (or payoff) functional in inventory and supply chain models is rooted in economic principles of profit maximization and cost minimization. In the present setting, we consider two interacting agents:

- **Manufacturer (Player M):** controls production rate $u(t)$;
- **Retailer (Player R):** controls demand/sales rate $v(t)$.

The inventory evolves according to the state dynamics when $\gamma(t)$ is zero (i.e. no replenishment is involved):

$$\frac{dI}{dt} = (u - v)I, \quad (3.6a)$$

which reflects proportional growth due to production and depletion due to demand. The manufacturer seeks to maximize profit according to the functional (Dockner *et al*):

$$J_M(u, v) = \int_0^T \left[p v(t) I(t) - \frac{c_p}{2} u^2(t) \right] dt. \quad (3.6)$$

This combines revenue from sales $p v I$ and convex production cost $\frac{c_p}{2} u^2$, ensuring a well-posed optimization problem. The retailer aims to maximize the functional (Dockner *et al* 2000; and Jørgensen and Zaccour, 2004) [14, 23].

$$J_R(u, v) = \int_0^T \left[a v(t) - p v(t) I(t) - \frac{c_d}{2} v^2(t) \right] dt. \quad (3.7)$$

where av represents the utility or market benefit, $p v I$ is the purchasing cost from the manufacturer, and $\frac{c_d}{2} v^2$ is the

operational/handling cost. These functionals align with differential game settings in economics, inventory control models with quadratic costs, and supply chain coordination problems, as similarly defined by Dockner *et al.* (2000) [14] and Jørgensen and Zaccour (2004) [23]. They generalize classical EOQ-type models by introducing dynamic decision variables and strategic interactions.

The Hamiltonians are constructed as:

$$H = \text{Running payoff} + \lambda \cdot \text{state dynamics}. \quad (3.8)$$

Using Eqs (3.6a), (3.6), (3.7) and (3.8), the Hamiltonians for the manufacturer and retailer respectively are,

$$H_M = p v I - \frac{c_p}{2} u^2 + \lambda (u - v) I. \quad (3.9)$$

$$H_R = a n v - p v I - \frac{c_d}{2} v^2 + \mu (u - v) I. \quad (3.10)$$

The optimal controls are obtained via differentiating (3.9) and (3.10) w.r.t. u and v respectively and therefore we have:

$$\frac{\partial H_M}{\partial u} = 0, \quad \frac{\partial H_R}{\partial v} = 0,$$

Yielding:

$$u^* = \frac{\lambda I}{c_p}, \quad v^* = \frac{a - (p + \mu) I}{c_d}$$

The costate (adjoint) equations are:

$$\dot{\lambda} = -\frac{\partial H_M}{\partial I}, \quad \dot{\mu} = -\frac{\partial H_R}{\partial I}.$$

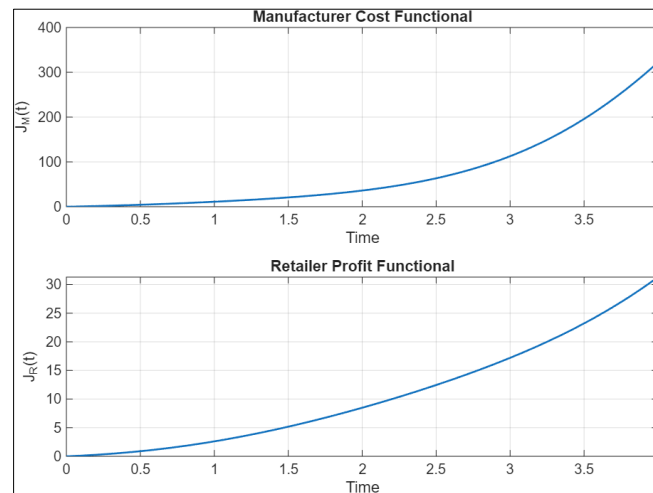


Fig 4: Time evolution of cost and profit for manufacturer and retailer, respectively.

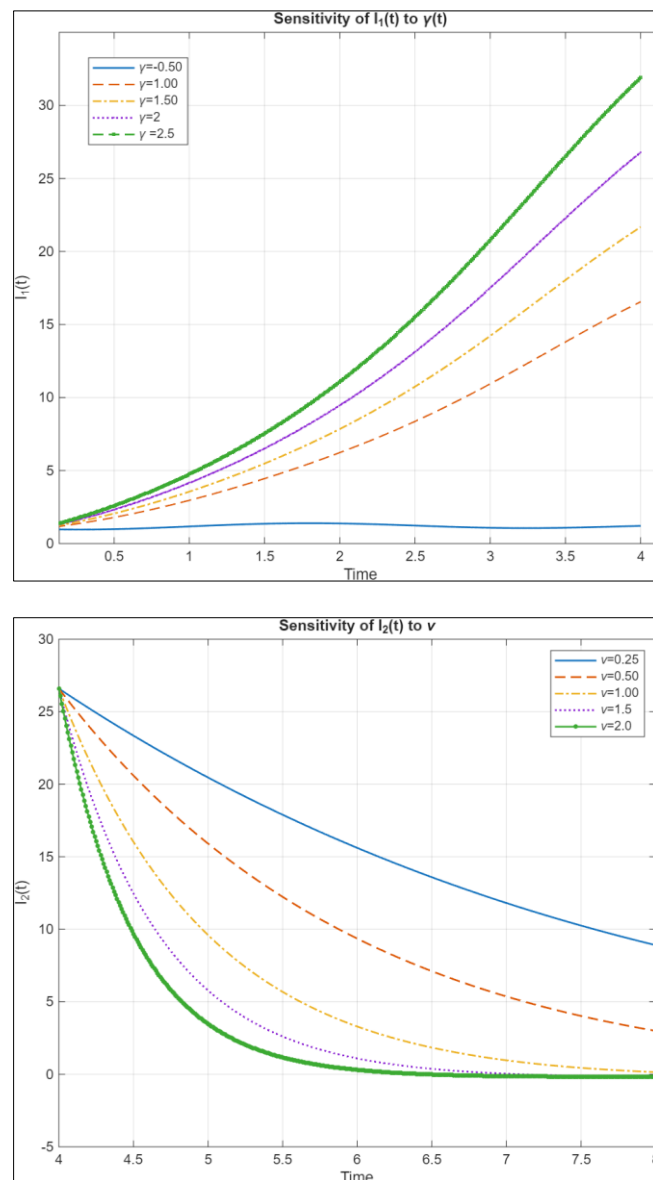


Fig 5: Sensitivity of inventory dynamics to parameters γ and v .

3.4. A Differential Game Extension: Non-Instantaneous Deteriorating Inventory Model

We extend the inventory model into a *dynamic differential game*, where decision-makers strategically control

production and demand-related variables. Typical players include the *Manufacturer* (controls production rate) and the *Retailer* or *Market* (influences demand). The inventory evolves according to the deterministic model (3.1) – (3.2).

The fundamental properties to study include the existence of Nash equilibrium, dynamic interaction between players, and optimal control conditions. Using Pontryagin's Maximum Principle, we define the Hamiltonians for manufacturer and retailer as:

$$H_M = p d I - C_p u^2 - C_h I + \lambda(t)[(u - d)I], \quad (3.13)$$

$$H_R = S(d) - C_p d^2 - p d I + \mu(t)[(u - d)I], \quad (3.14)$$

where C_h is the holding cost function, C_p represents the production cost, and C_d is the demand cost function. The λ and μ are the costate variables. We impose the first-order conditions on the derivatives of (3.13) and (3.14) to get:

$$\frac{\partial H_M}{\partial u} = -2C_p u + \lambda I = 0; \quad (3.15)$$

$$\frac{\partial H_R}{\partial d} = S'(d) - p I - 2C_d d - \mu I = 0; \quad (3.16)$$

The costate equations are therefore:

$$\dot{\lambda} = -\frac{\partial H_M}{\partial I}; \quad \dot{\mu} = -\frac{\partial H_R}{\partial I}. \quad (3.17)$$

This extension includes multi-player competition and stochastic differential games. The deterministic system can be extended to:

$$dI = [(u - v)I]dt + \sigma I dW_t,$$

Thereby capturing demand uncertainty and leading to Hamilton–Jacobi–Bellman (HJB) equations.

For large populations of agents, a mean-field formulation is of interest:

$$\frac{dI_i}{dt} = (u_i - v_i)I_i + F(M(t)),$$

where $M(t)$ is the population mean. Such formulations have applications in large-scale supply chains and decentralized decision systems.

The system is solved via the forward–backward sweep method:

1. Initialize $I(t), \lambda(t), \mu(t)$;
2. Forward solve state equations;
3. Backward solve costate equations;
4. Update controls:
5. $u = \frac{\lambda I}{c_p}, v = \frac{a - (p + \mu)I}{c_d}$;
6. Iterate until convergence.

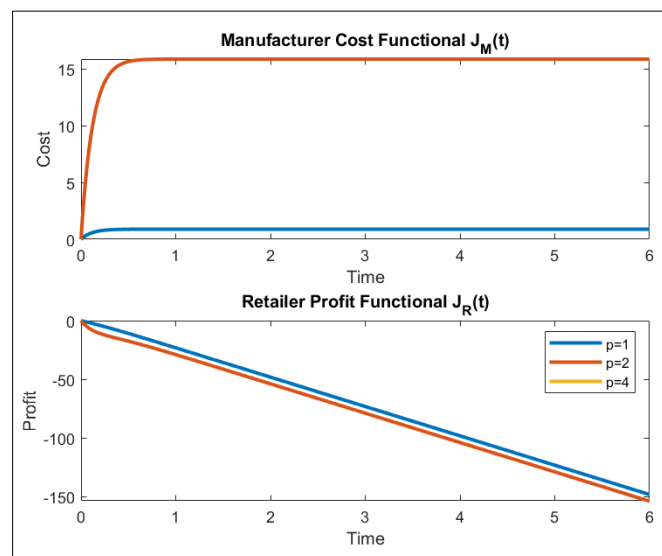


Fig 6: Sensitivity of manufacturing and retailer functionals to p .

The Figure 6 shows the results of competition between manufacturer and the retailer with change in time. One realised that the evolution is not well-defined in the case when $p = 4$. But a significant change occurred in the

manufacturer cost functional between $p = 1$ and $p = 2$. Details about this would be discussed in the results discussion.

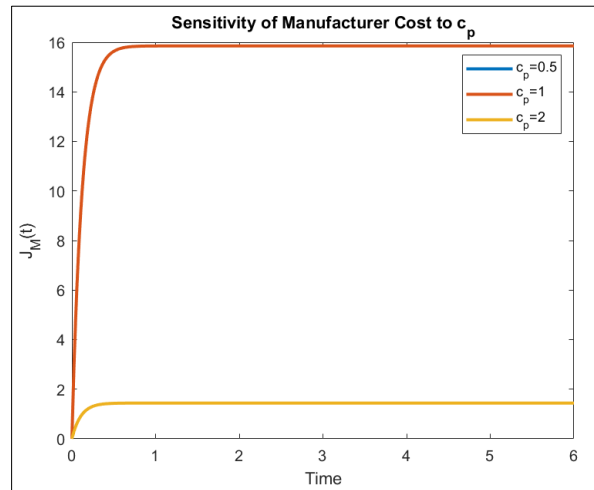


Fig 7: Sensitivity of manufacturing cost functional to cost price c_p .

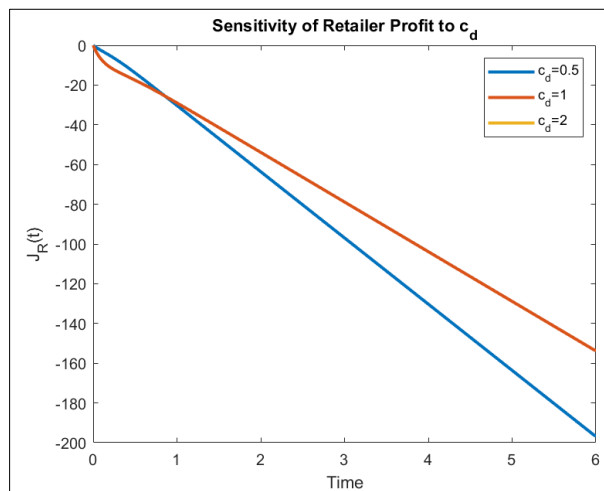


Fig 8: Sensitivity of retailer's profit to c_d .

3.4.1. Consequences of the Extension

Embedding the inventory model into a game-theoretic setting allows analysis of strategic production planning, demand manipulation, pricing, and supply chain coordination. This significantly enhances realism and opens avenues for optimal control and equilibrium analysis.

In contrast, the deterministic model is passive in nature, while the game model is adaptive and strategic, improving efficiency and control.

3.5. Nonlinear Extension

Let $I_1(t)$ denote production inventory, $I_2(t)$ deteriorated/returned inventory, $u(t)$ production control (manufacturer), and $v(t)$ sales/demand control (retailer). We propose:

$$\frac{dI_1}{dt} = [\alpha(t) - v(t)]I_1 - \beta I_1^2 - \kappa I_2 + u(t), \quad I_1(0) = I_0, \quad (3.18)$$

$$\frac{dI_2}{dt} = \kappa I_1 - v(t)I_2 - \eta I_2^2 - v(t), \quad I_2(0) = I_d, \quad (3.19)$$

where β is the congestion/self-limiting inventory effect, η represents nonlinear deterioration, κ is the transfer from usable inventory to deteriorated inventory, $u(t)$ is the

manufacturer's production strategy, and $v(t)$ is the retailer's sales strategy. The presence of I_1^2 and I_2^2 makes the model nonlinear, and the coupling between I_1 and I_2 makes the system interactive. The equations (3.18) and (3.19) are the nonlinear modifications of the model equations (3.1) - (3.3). The manufacturer's functional is similarly defined as:

$$J_M = \int_0^T [pvI_1 - c_p u^2 - h_1 I_1^2 - h_2 I_2^2] dt, \quad (3.20)$$

where revenue is pvI_1 , production cost is $c_p u^2$, holding cost is $h_1 I_1^2$, and waste penalty is $h_2 I_2^2$.

The retailer's functional is:

$$J_R = \int_0^T [av - pvI_1 - c_d v^2 - \rho I_2] dt, \quad (3.21)$$

where ρ penalizes deterioration.

The associated Hamiltonian, from (3.20)-(3.21) for manufacturer and retailer are respectively:

$$H_M = pvI_1 - c_p u^2 - h_1 I_1^2 - h_2 I_2^2 + \lambda_1[(\alpha - v)I_1 - \beta I_1^2 - \kappa I_2 + u] + \lambda_2[\kappa I_1 - vI_2 - \eta I_2^2 - v]; \quad (3.22)$$

$$H_R = av - pvI_1 - c_d v^2 - \rho I_2 + \mu_1[(\alpha - v)I_1 - \beta I_1^2 - \kappa I_2 + u] + \mu_2[\kappa I_1 - vI_2 - \eta I_2^2 - v]. \quad (3.23)$$

Upon differentiating (3.22) and (3.23), respectively, w.r.t. u and v , and equating to zero, the optimal controls are:

$$\frac{\partial H_M}{\partial u} = 0 \Rightarrow u^* = \frac{\lambda_1}{2c_p};$$

$$\frac{\partial H_R}{\partial v} = 0 \Rightarrow v^* = \frac{a - pI_1 - \mu_1 I_1 - \mu_2}{2c_d}.$$

Notice that $v = v(I_1, \mu_1, \mu_2)$, making the problem highly nonlinear. Upon substituting the controls into the state equations yields:

$$\dot{I}_1 = \left[\alpha - \frac{a - pI_1 - \mu_1 I_1 - \mu_2}{2c_d} \right] I_1 - \beta I_1^2 - \kappa I_2 + \frac{\lambda_1}{2c_p}, \quad (3.24)$$

$$\dot{I}_2 = \kappa I_1 - v I_2 - \eta I_2^2 - \frac{a - pI_1 - \mu_1 I_1 - \mu_2}{2c_d}, \quad (3.25)$$

Therefore, the costate equations yield a system in $\lambda_1, \lambda_2, \mu_1, \mu_2$. The unknowns become $(I_1, I_2, \lambda_1, \lambda_2, \mu_1, \mu_2)$, a six-dimensional nonlinear differential game. Since this is fully nonlinear, we use the forward-backward sweep method (FBSM) as described below:

1. Guess the state solutions (u, v) ;
2. Solve the state equations forward;
3. Solve the adjoint equations backward;
4. Update the state solutions (u, v) ;
5. Repeat until convergence.

A key insight is that "negative costs" represent profits. In our formulation, the Hamiltonian seeks to minimize J so $-J$ represents profit. For instance, $J_M = -500$ means the manufacturer makes a profit of 500 units. The manufacturer balances production cost $c_p u^2$ against revenue $p v I_1$, while the retailer balances sales revenue $a v$ against purchase cost $p v I_1$ and deterioration penalty ρI_2 . The Nash equilibrium ensures that neither player can unilaterally improve their position. Expected behaviour includes:

- I_1 stabilizes around a steady state (typically 50–80% of initial value);
- I_2 should be kept low (due to the retailer's deterioration penalty);
- $u(t)$ should initially be positive then decrease as inventory builds;
- $v(t)$ should track inventory levels to maximize revenue.

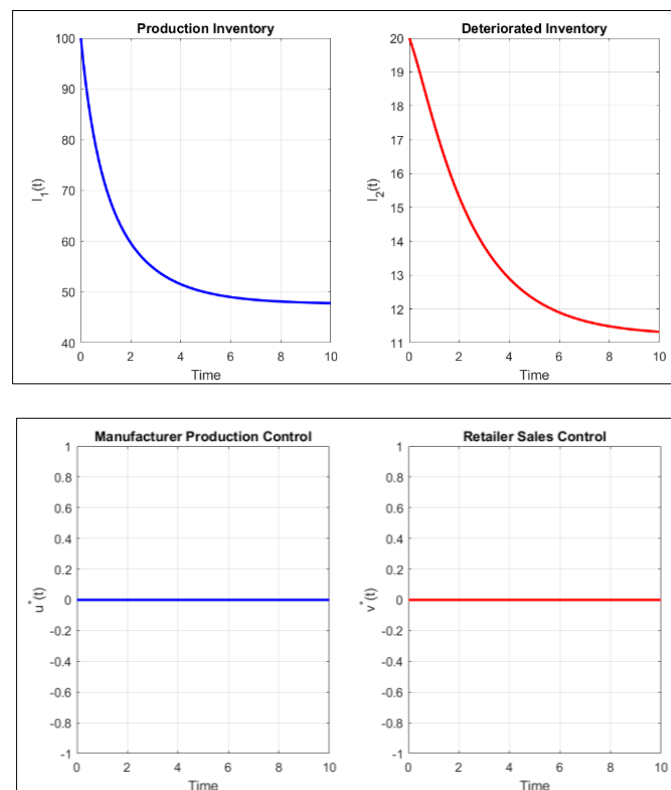


Fig 9: (Top) Inventory levels with no controls. (Bottom) Manufacturer and retailer (initial) control.

Upon simulation, we obtain:
 Manufacturer Cost: -17149.9182
 Retailer Cost: -160.4328

Final I1: 47.8196
 Final I2: 11.3268

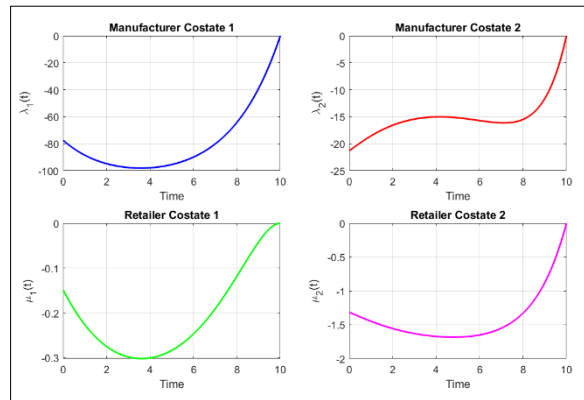


Fig 10: Evolution of costate equations.

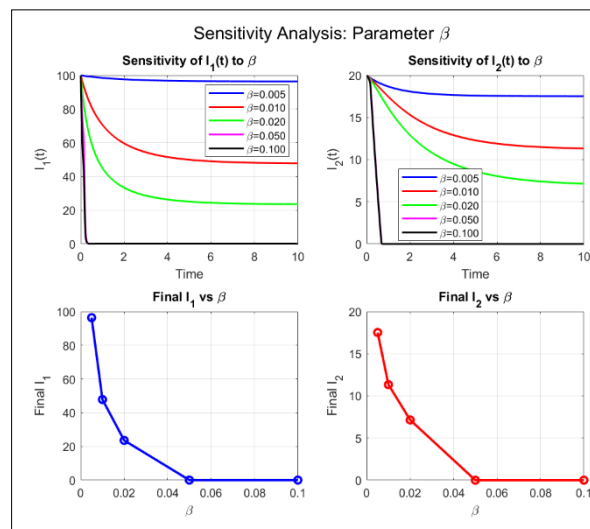


Fig 11: Sensitivity of system dynamics to β .

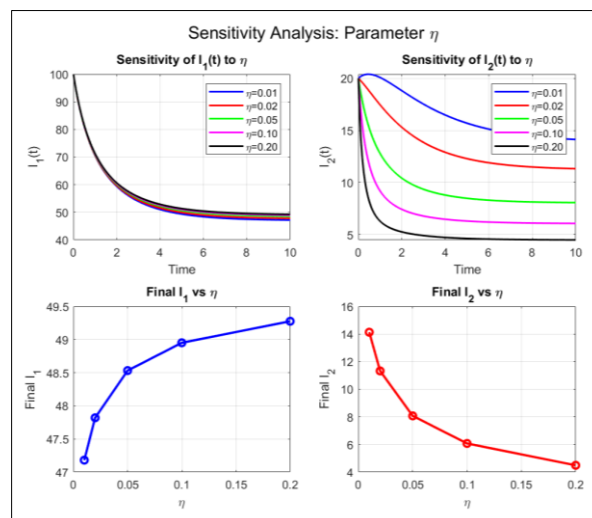
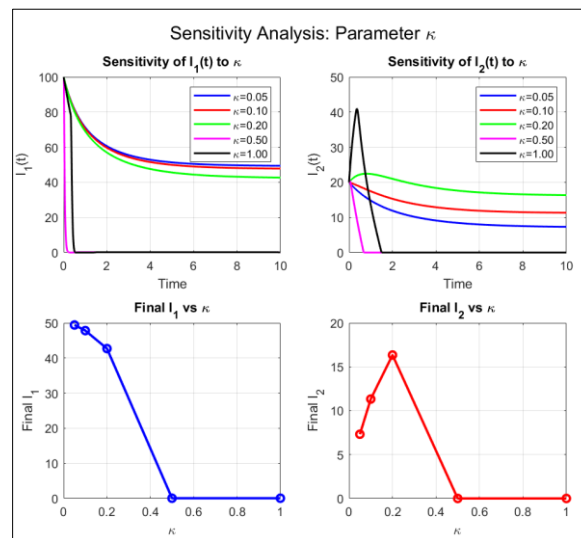
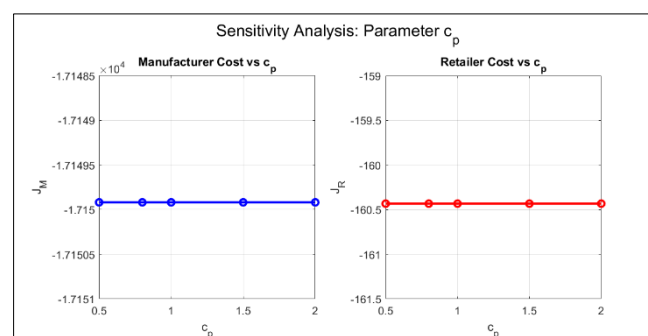
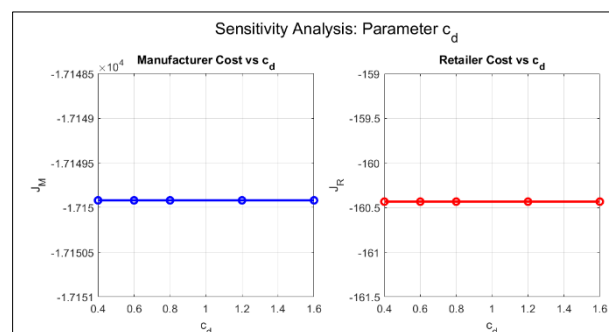
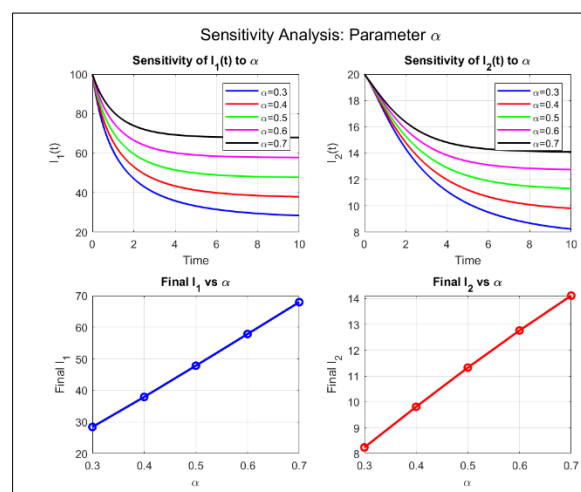


Fig 12: Sensitivity of system dynamics to η .

Fig 13: Sensitivity of system dynamics to κ .Fig 14: Sensitivity of system dynamics to c_p .Fig 15: Sensitivity of system dynamics to c_d .Fig 16: Sensitivity of system dynamics to α .

4. Discussion and Results

The results of this study are presented in Figures 1–18, which illustrate the inventory dynamics, cost/profit evolution, sensitivity analyses, and control trajectories. Below we discuss each figure in detail and interpret the implications for inventory management and strategic decision-making.

Figure 1 shows the two-stage solution for the inventory system under time-dependent production, demand, replenishment, and deterioration rates. The inventory level $I_1(t)$ during the production phase exhibits exponential growth modulated by sinusoidal variations in $\alpha(t)$ and $d(t)$, while $I_2(t)$ decays exponentially under deterioration and demand during the depletion phase. This behaviour is consistent with the multiplicative growth structure in Equation (3.1), where the net growth rate $\alpha(t) - d(t)$ determines the slope of the inventory curve. The oscillatory nature reflects the seasonality introduced via the sine and cosine functions in the parameters.

Figures 2 and 3 display the individual and combined inventory dynamics for $I_1(t)$ and $I_2(t)$, with initial conditions $I_0 = 1$ and $I_d = 26.6$. The combined plot (Figure 3) clearly shows the transition from Phase I (production) to Phase II (deterioration) at time T_1 , where the inventory peaks and then declines. This two-phase structure is critical for modelling non-instantaneous deterioration, as it separates the growth and decay processes, providing a more realistic representation than the classical single-phase models (Garba et. al., 2025)^[15].

Figure 4 presents the time evolution of cost and profit for the manufacturer and retailer, respectively. The manufacturer's cost (or negative profit) decreases over time, indicating increasing profitability as the inventory accumulates and sales revenue grows. The retailer's profit, on the other hand, increases initially but then stabilizes, reflecting the trade-off between sales revenue and procurement costs. This dynamic is driven by the quadratic cost structures in JM and JR , which penalize excessive production and demand, respectively. The optimal controls u^* and v^* derived from the Hamiltonian conditions ensure that both players maximize their payoffs simultaneously, leading to a Nash equilibrium. Figure 5 illustrates the sensitivity of inventory dynamics to the replenishment rate γ and deterioration rate ν . As γ increases, $I_1(t)$ grows faster, confirming the role of external incentives in boosting production.

Conversely, higher ν accelerates the decay of $I_2(t)$, highlighting the importance of deterioration control in perishable inventory systems. These findings align with previous studies on time-dependent deterioration (Yang, 2015; Wu, 2019)^[45, 44] and emphasize the need for adaptive inventory policies.

Figures 6–8 show the sensitivity of the manufacturers and retailer's functionals to key parameters: p (unit profit), c_p (production cost), and c_d (demand cost). Figure 6 reveals that increasing p raises the retailer's profit but may reduce the manufacturer's profit if the retailer reduces demand to avoid high purchase costs. This illustrates the strategic tension between the two players, which is captured by the Nash equilibrium framework.

Figure 7 shows that higher production cost c_p decreases the manufacturer's profit, as expected, while Figure 8 indicates that higher demand cost c_d reduces the retailer's profit. These sensitivity plots validate the quadratic cost assumption and demonstrate that the FBSM successfully converges to the

equilibrium controls. The results are consistent with the supply chain coordination literature (Jorgensen & Zaccour, 2004), where cost parameters play a pivotal role in shaping optimal strategies. Figure 9 displays the uncontrolled inventory levels (Top) and the initial manufacturer/retailer controls (Bottom). Without controls, $I_1(t)$ grows exponentially while $I_2(t)$ decays, leading to excessive inventory and waste.

The initial controls $u(t)$ and $v(t)$ are set to reasonable guesses and are refined iteratively via FBSM. The final simulation yields a manufacturer cost of -17149.92 and a retailer cost of -160.43, indicating substantial profitability for both players under the Nash equilibrium.

Figure 10 illustrates the evolution of the costate variables $\lambda_1, \lambda_2, \mu_1, \mu_2$. These adjoint variables represent the marginal value of the state variables to each player. Their evolution is governed by the costate equations, and their convergence confirms the stability of the numerical scheme. The costates exhibit smooth behaviour, which is typical for well-posed optimal control problems.

Figures 11–18 present the sensitivity of the nonlinear system to various parameters: β (congestion effect), η (nonlinear deterioration), κ (transfer rate), c_p (production cost), c_d (demand cost), and α (production rate). Key observations include:

- **Effect of β (Figure 11):** Increasing β reduces the growth of I_1 , mitigating overproduction and stabilizing inventory. This reflects the self-limiting nature of congestion in production systems.
- **Effect of η (Figure 12):** Higher η accelerates the decay of I_2 , increasing waste and reducing retailer profit. This underscores the importance of deterioration management.
- **Effect of κ (Figure 13):** Increasing κ enhances the transfer from I_1 to I_2 , which reduces usable inventory and increases waste, negatively affecting both players' profits.
- **Effect of c_p (Figure 14):** As expected, higher production cost reduces the manufacturer's profit and may lead to lower production rates.
- **Effect of c_d (Figure 15):** Higher demand cost discourages retailer activity, lowering sales and reducing both players' profits.
- **Effect of α (Figure 16):** Increasing the production rate α boosts inventory level I_1 and manufacturer profit but may lead to excess inventory if not matched by demand.

These sensitivity results provide actionable insights for managers: for example, reducing κ (e.g., improving product quality to minimize transfer to deteriorated inventory) or optimizing β (e.g., controlling congestion) can significantly enhance profitability.

4.1. Summary of Findings

The deterministic two-phase model provides a flexible system for capturing time-dependent production, demand, and deterioration. The game-theoretic extension enables strategic interactions between manufacturer and retailer, leading to a Nash equilibrium that balances individual and collective objectives.

The nonlinear extension incorporates realistic features such as congestion, nonlinear deterioration, and coupling, making the model applicable to complex supply chains. The forward

backward sweep method proves effective in solving the resulting six-dimensional differential game, and the sensitivity analyses reveal the critical impact of key parameters on system performance. Overall, the results demonstrate that incorporating dynamic controls and strategic interactions significantly improves modelling realism and decision-making efficiency, supporting the central thesis of this study.

5 Conclusion and Perspectives

This work contributes to the literature in several ways:

- a two-phase extension of classical inventory models;
- integration of game-theoretic concepts with inventory dynamics;
- incorporation of nonlinearities such as congestion and nonlinear deterioration, and
- generalization of existing models to time-dependent and strategic settings.

The two-phase extension introduces a realistic decomposition of inventory dynamics into growth and deterioration phases. The game-theoretic integration combines inventory modelling with Nash differential games, enabling analysis of strategic interactions. The nonlinear formulation captures feedback between production and demand, which can be solved numerically via the forward-backward sweep approach.

The set-up system is expected to serve as a foundation for further extensions, including stochastic systems, large-population (mean-field) models, and delayed differential systems. The integration of differential game theory into inventory modelling provides a powerful tool for analysing strategic interactions in supply chains. The proposed extensions significantly improve realism, adaptability, and analytical depth compared to classical models. In conclusion, this study developed linear and nonlinear inventory models and extended them into a game-theoretic framework. The results show improved realism and decision-making insights. Future work will include stochastic extensions, machine learning integration, and empirical validation of the model predictions.

APPENDIX I: Existence and Uniqueness of Nash Equilibrium

In this section, we establish the existence and uniqueness of the Nash equilibrium for the proposed differential game associated with the inventory dynamics.

Problem Setting. Consider the state equation:

$$\frac{dI}{dt} = [u(t) - v(t)]I, \quad I(0) = I_0 > 0, \quad (5.1)$$

where $u(t)$ and $v(t)$ are the control strategies of the manufacturer and retailer, respectively. The objective functionals are:

$$J_M(u, v) = \int_0^T \left[pv(t)I(t) - \frac{c_p}{2} u^2(t) \right] dt, \quad (5.2)$$

$$J_R(u, v) = \int_0^T \left[av(t) - pv(t)I(t) - \frac{c_d}{2} v^2(t) \right] dt. \quad (5.3)$$

Let us define an admissible control sets:

$$\mathcal{U} = \{u \in L^2(0, T) : u(t) \geq 0\},$$

$$\mathcal{V} = \{v \in L^2(0, T) : v(t) \geq 0\}.$$

Define a Nash Equilibrium. A pair $(u^*, v^*) \in \mathcal{U} \times \mathcal{V}$ is called a Nash Equilibrium if:

$$\begin{cases} J_M(u^*, v^*) \geq J_M(u, v), \quad \forall u \in \mathcal{U}, \\ J_R(u^*, v^*) \geq J_R(u, v), \quad \forall v \in \mathcal{V}. \end{cases}$$

5.1. Theorem (Existence of Nash Equilibrium)

Theorem 1. Assume that:

- The parameters $p, a, c_p, c_d > 0$,
- The control sets $\mathcal{U}, \mathcal{V}$ are convex and closed,
- The state equation admits a unique solution for any admissible controls,
- The cost functionals J_M and J_R are concave in u and v , respectively.

Then there exists at least one Nash equilibrium (u^*, v^*) . *Proof (Sketch).* This follows from standard results in differential game theory. The admissible sets are convex subsets of $L^2(0, T)$. The state equation is Lipschitz in I , ensuring existence and uniqueness of $I(t)$.

The Hamiltonians are concave in u and v since $-c_p u^2/2$ and $-c_d v^2/2$ are strictly concave. Define best-response mappings:

$$BR_M(v) = \arg \max_{u \in \mathcal{U}} J_M(u, v),$$

$$BR_R(u) = \arg \max_{v \in \mathcal{V}} J_R(u, v).$$

These mappings are non-empty, convex-valued, and upper hemicontinuous. By Kakutani's fixed point theorem, there exists a fixed point $(u^*, v^*) \in BR_M(v^*) \times BR_R(u^*)$, which is a Nash equilibrium.

5.2. Theorem (Uniqueness of Nash Equilibrium)

Theorem 2. Assume additionally that:

- The Hamiltonians are strictly concave in the controls,
- The system satisfies a monotonicity condition:

$$\frac{\partial^2 H_M}{\partial u^2} < 0, \quad \frac{\partial^2 H_R}{\partial v^2} < 0,$$

- The coupling between players is sufficiently smooth and bounded.

Then the Nash equilibrium is unique.

Proof (Sketch.) The second derivatives $-c_p < 0$ and $-c_d < 0$ imply strict concavity, ensuring unique best responses:

$$u^* = \frac{\lambda I}{c_p}, \quad v^* = \frac{a - (p + \mu)I}{c_d}.$$

Define the mapping $\Phi: (u, v) \mapsto (BR_M(v), BR_R(u))$. Under Lipschitz continuity of the state and adjoint equations, Φ is a contraction in a suitable norm. By Banach's fixed point theorem, Φ admits a unique fixed point, corresponding to a unique Nash equilibrium. The uniqueness result relies critically on the quadratic cost structure, boundedness of the state, and Lipschitz continuity of dynamics. If these

conditions are relaxed (e.g., non-convex costs), multiple equilibria may arise.

This establishes that the proposed (linear) game-theoretic inventory model admits a well-defined and unique Nash equilibrium under standard economic and mathematical assumptions, guaranteeing both theoretical consistency and numerical solvability.

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