



Sandwich Results for Certain Differential Operator on Multivalent functions Operator

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Abstract

The aim of the present paper is to prove the existence of the sandwich theorem for differential operator defined by the convolution of univalent holomorphic function with the distribution probability function in simple – connected domain represented by the unit disk $\mathcal{U} = \{s = s \in \mathbb{C}: |z| < 1\}$ by obtaining results that preserving the subordination and super ordination properties.

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1. Introduction

We denote by $\mathbb{H}(\mathcal{U})$ be the class of analytic functions in the open unit disc $\mathcal{U} = \{z \in \mathbb{C}: |z| < 1\}$ and let $\mathbb{H}[\nu, n]$ be the subclass of $\mathbb{H}(\mathcal{U})$ including from-specific functions of the form $f(z) = \nu + \nu_n z^n + \nu_{n+1} z^{n+1} + \dots$, with $\mathbb{H}_0 = \mathbb{H}[0, 1]$ and $\mathbb{H} = \mathbb{H}[1, 1]$. Let $\mathcal{A}(\ell)$ indicate the class of functions of the form

$$h(z) = z^\ell + \sum_{n=1}^{\infty} \nu_{\ell+n} z^{\ell+n} \quad (\ell \in \mathbb{N} = \{1, 2, 3, \dots\}; z \in \mathcal{U}) \quad (1.1)$$

which are analytic and univalent the open unit disc $\mathcal{U} = \{z \in \mathbb{C}: |z| < 1\}$.

Let $h(z)$ and $\mathcal{F}(z)$ are two analytic functions in $\mathbb{H}(\mathcal{U})$. We say that the function $h(z)$ is subordinate [14] to $\mathcal{F}(z)$ or $\mathcal{F}(z)$ is a superordinate to $h(z)$ in such a case we write $h(z) < \mathcal{F}(z)$ if there exist analytic function $\mathcal{B}(z)$ in $\mathcal{U}$ with the properties $\mathcal{B}(0) = 0$ and $|\mathcal{B}(z)| < 1 (z \in \mathcal{U})$. such that $h(z) = \mathcal{F}(\mathcal{B}(z))$. If $\mathcal{F}(z)$ is univalent, then $h(z) < \mathcal{F}(z)$ if and only if $h(0) = \mathcal{F}(0)$ and $h(\mathcal{U}) \subset \mathcal{F}(\mathcal{U})$. Let $\Psi: \mathbb{C}^3 \times \mathcal{U} \rightarrow \mathbb{C}$ and suppose that the function $\mathcal{K}(z)$ is univalent in $\mathcal{U}$. If the function $\mathcal{T}(z)$ is analytic in $\mathcal{U}$ and satisfies the following first order differential subordination:

$$\Psi(\mathcal{T}(z), z\mathcal{T}'(z); z) < \mathcal{K}(z) \quad (1.2)$$

then $\mathcal{T}(z)$ is called a solution of the differential subordination (1.2). Furthermore, a given univalent function $\mathfrak{B}(z)$ is said to as a dominant of the solutions of (1.2), or, more simply, a dominant if $\mathcal{T}(z) < \mathfrak{B}(z)$ for all $\mathcal{T}(z)$ satisfying (1.2). A univalent dominant $\tilde{\mathfrak{B}}(z)$ that satisfies $\tilde{\mathfrak{B}}(z) < \mathfrak{B}(z)$ given any dominants $\mathcal{T}(z)$ of (1.2) is acknowledged as the best dominant. Assuming that $\mathcal{T}(z)$ and $\Psi(\mathcal{T}(z), z\mathcal{T}'(z); z)$ are univalent in $\mathcal{U}$ and if $\mathcal{T}(z)$ satisfies the following first order differential superordinate:

$$\mathcal{K}(z) < \Psi(\mathcal{T}(z), z\mathcal{T}'(z); z). \quad (1.3)$$

then $\mathcal{T}(z)$ is solutions of the differential subordination (1.3). An analytic function $\mathfrak{B}(z)$ is called subordination of the solutions of the differential subordination (1.3) if $\mathfrak{B}(z) < \mathcal{T}(z)$ for all $\mathcal{T}(z)$ satisfying (1.3). A univalent subordinate $\tilde{\mathfrak{B}}$ that satisfies $\mathfrak{B} < \tilde{\mathfrak{B}}$ for all subordinants of (1.3) is called the best subordinate see [1.2]).

To obtain the results that we want to document in the research, we will need the definitions and basic lemmas as a preliminary result.

Definition 1. Let $\mathcal{W}_{z,\ell(n)}^{\mu,n}: \mathcal{A}(\ell) \rightarrow \mathcal{A}(\ell)$ be a new differential operator defined by

$$\mathcal{W}_{z,\ell(n)}^{\mu} h(z) = \frac{\Gamma(\ell+1)}{\Gamma(\ell-\mu+1)} z^{\ell-\mu} + \sum_{k=\ell+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\mu+1)} \left(\frac{k+\lambda}{\ell+\lambda}\right)^n \nu_k z^k, \quad$$

$$(\lambda > -\ell; \mu \in \mathbb{R}; \ell \in \mathbb{N})$$

$$2.41) \text{ If } \mu = n = 0$$

we have

$$\mathcal{W}_{z,\ell(0)}^0 h(z) = h(z).$$

The following useful identities are satisfied by the operator

$$z \left(\mathcal{W}_{z,\ell(n)}^{\mu} h(z) \right)' = (\ell - \mu + \lambda) \mathcal{W}_{z,\ell(n+1)}^{\mu} h(z) - (\lambda) \mathcal{W}_{z,\ell(n)}^{\mu} h(z).$$

Definition 2. Denote by $\mathbb{F}$ the set of all functions $\mathfrak{B}(z)$ that are analytic and injective on $\overline{\mathcal{U}} \setminus \mathcal{E}(\mathfrak{B})$ where

$$\mathcal{E}(\mathfrak{B}) = \left\{ \xi \in \partial \mathcal{U} : \lim_{z \rightarrow \xi} \mathfrak{B}(z) = \infty \right\}.$$

and are such that $\mathfrak{B}'(\xi) \neq 0$ for $\xi \in \partial \mathcal{U} \setminus \mathcal{E}(\mathfrak{B})$. Further let the subclass of $\mathbb{F}$ for which $\mathfrak{B}(0) = a$ be denoted by $\mathbb{F}(a)$. $\mathbb{F}(0) = \mathbb{F}_0$ and $\mathbb{F}(1) \equiv \mathbb{F}$.

Definition 3. A function $\mathcal{L}(z, t)$ ($z \in \mathcal{U}, t \geq 0$) is said to be a subordination chain if $\mathcal{L}(0, 1)$ is analytic and univalent in $\mathcal{U}$ for all $t \geq 0$. $\mathcal{L}(z, 0)$ is continuously differentiable on $[0, 1)$ for all $z \in \mathcal{U}$ and $\mathcal{L}(z, t_1) < \mathcal{L}(z, t_2)$ for all $0 \leq t_1 \leq t_2$.

Lemma 1. The function $\mathcal{L}(z, t): \mathcal{U} \times [0; 1) \rightarrow \mathbb{C}$ of the form

$$\mathcal{L}(z, t) = \nu_1(t)z + \nu_2(t)z^2 + \cdots (\nu_1(t) \neq 0; t \geq 0).$$

And $\operatorname{Re} \left\{ \frac{z \partial \mathcal{L}(z, t) / \partial z}{\partial \mathcal{L}(z, t) / \partial t} \right\} > 0$ ($z \in \mathcal{U}, t \geq 0$) if and only if $\lim_{t \rightarrow \infty} |\nu_1(t)| = \infty$ is a subordination chain.

Lemma 2. Suppose that the function $\mathbb{H}: \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfies the condition

$$\operatorname{Re}\{\mathbb{H}(is; t)\} \leq 0$$

for all real s and for all $t \leq -\ell(1 + s^2)/2$. $\ell \in \mathbb{N}$. If the function $\mathcal{T}(z) = 1 + \mathcal{T}_{\ell} z^{\ell} + \mathcal{T}_{\ell+1} z^{\ell+1} + \cdots$ is analytic in $\mathcal{U}$ and

$$\operatorname{Re}\{\mathbb{H}(\mathcal{T}(z); z\mathcal{T}'(z))\} > 0 \quad (z \in \mathcal{U}).$$

Then $\operatorname{Re}\{\mathcal{T}(z)\} > 0$ for $z \in \mathcal{U}$.

Lemma 3. Let $x, d \in \mathbb{C}$ with k not equal zero and let $\mathcal{K} \in \mathbb{H}(\mathcal{U})$ with $\mathcal{K}(0) = c$. If $\operatorname{Re}\{k\mathcal{K}(z) + \gamma\} > 0$ ($z \in \mathcal{U}$), then the analytic in $\mathcal{U}$ that satisfying satisfies $\operatorname{Re}\{x\mathcal{K}(z) + d\} > 0$ for $z \in \mathcal{U}$ represent a solution of the following differential equation:

$$\mathfrak{B}(z) + \frac{z\mathfrak{B}'(z)}{x\mathfrak{B}(z)+d} = \mathcal{K}(z) \quad (z \in \mathcal{U}; \mathfrak{B}(0) = c).$$

Lemma 4. Let $\mathcal{T} \in \mathbb{F}(\nu)$ and let $\mathfrak{B}(z) = \nu + \nu_{\ell} z^{\ell} + \nu_{\ell+1} z^{\ell+1} + \cdots$ be analytic in $\mathcal{U}$ with $\mathfrak{B}(z) \neq \nu$ and $\ell \geq 1$. If $\mathfrak{B}$ is not subordinate to $\mathcal{T}$, then there exists two points $z_0 = r_0 e^{i\theta} \in \mathcal{U}$ and $\xi_0 \in \partial \mathcal{U} \setminus \mathcal{E}(\mathfrak{B})$ such that $\mathfrak{B}(\mathcal{U}_{r_0}) \subset \mathcal{T}(\mathcal{U})$; $\mathfrak{B}(z_0) = \mathcal{T}(\xi_0)$ and $z_0 \mathcal{T}'(z_0) = \nu \xi_0 \mathcal{T}'(\xi_0)$ ($\nu \geq \ell$).

Lemma 5. Let $\mathfrak{B} \in \mathbb{H}[\nu, 1]$ and $\Psi: \mathbb{C}^2 \rightarrow \mathbb{C}$. Also set $\Psi(\mathfrak{B}(z), z\mathfrak{B}'(z)) = \mathcal{K}(z)$. If $\mathcal{L}(z, t) = \Psi(\mathfrak{B}(z), tz\mathfrak{B}'(z))$ is a

subordination chain and $\mathfrak{B} \in \mathbb{H}[\nu, 1] \cap \mathbb{F}(\nu)$. then

$$\mathcal{K}(z) < \Psi(\mathcal{T}(z). z\mathcal{T}'(z))$$

Therefore

$$\mathfrak{B}(z) < \mathcal{T}(z).$$

Moreover, $\mathfrak{B}$ is the best subordinant if $\mathfrak{B}$ is univalent solution belonging to $\mathbb{F}(\nu)$ for the differential equation $\Psi(\mathfrak{B}(z). z\mathfrak{B}'(z)) = \mathcal{K}(z)$

The aim of this study, we examine certain properties that preserve subordination and super ordination related integral operator $\mathcal{W}_{z, \ell(n)}^{\mu} h(z)$. By using the differential operator mentioned above, a conclusion will be drawn about the quality of the sandwich result.

2. Sandwich theorem of functions defined by operator concerning with Bell distribution

Ala. n. Mourah and Jamul Salah [Geometric properties and Neighbor hands of certain sub class of analytic functions defined by using Bell Distribution, contemporary Mathematics, Do1: [http://doi-org/ 10.37256/cm.5420245425](http://doi-org/10.37256/cm.5420245425), (2024), pp.1-10.]

Using the probability density function of a discrete random variable x that follows the bell distribution expressed as

$$p(x = c) = \frac{\beta^c e^{(-\beta^2)+1} \lambda_c}{c!}, c=1,2,3, \dots$$

Where $\lambda_c = \frac{1}{e} \sum_{x=0}^{\infty} \frac{x^c}{x!}$ Is the Bell numbers? $c \geq 2, \beta > 0$.

To introduce a new power series, utilizing bell distribution probabilities as coefficients

$$I(\beta, s) = s + \sum_{c=k+1}^{\infty} \frac{\beta^{c-1} e^{(-\beta^2)+1} \lambda_c}{(c-1)!} \cdot s \in \mathcal{U}, \beta > 0. \quad (1.1)$$

For a function k belong in the collection of all holomorphic functions that defined in a disk $\mathcal{U}$ as a simple connected domain that given by

$$k(s) = s + \sum_{c=k+1}^{\infty} \delta_c z^k \quad (s \in \mathcal{U}, \delta_c > 0, k \in \mathbb{N}) \quad (1.2)$$

The operator $\mathcal{D}k(s): \mathcal{A} \rightarrow \mathcal{A}$ as

$$\mathcal{D}k(\beta, \lambda, s)k(s) = I(\beta, s) + k(s) = s + \sum_{c=k+1}^{\infty} \frac{\beta^{c-1} e^{(-\beta^2)+1} \lambda_c}{(c-1)!} \delta_c s^c. \text{ We define anew operator } \beta^m(\alpha_1 \alpha_2, \beta, \lambda)k(s) = (L^m(\delta, \omega))(s) * \mathcal{D}(\beta, \lambda_c)k(s)$$

Where $Q^m(\alpha_1 \alpha_2)(s)$ is analytic function defined as?

$$\beta^m(\omega, \beta, \lambda_c)k(s) = s + \sum_{c=k+1}^{\infty} \frac{\beta^{c-1} e^{(-\beta^2)+1} \lambda_c}{(c-1)!} \left(\frac{\omega}{\omega+n-1} \right) \delta_c s^c. \quad k = \{1, 2, \dots\}, k \in \mathbb{N}$$

Where

$$Re(\omega) > 0, m \geq 1, \lambda_c = \frac{1}{e} \sum_{x=0}^{\infty} \frac{x^c}{x!} \text{ Is the bell numbers, } \beta > 0$$

By simple computation, we can obtain that

$$\begin{aligned} & (1 + \bar{\omega})(\beta^{m+1}(\omega, \beta, \lambda_c)k(s)) \\ &= e^{e^{-\beta^2+1}} s(\beta^m(\omega, \beta, \lambda_c)k(s) + (1 + \bar{\omega} - e^{e^{-\beta^2+1}})(\beta^{m+1}(\omega, \beta, \lambda_c)k(s)) \end{aligned} \quad (1.3)$$

[Nak Eun Cho, strong differential subordination properties for analytic functions involving the komatu integral operator] cho boundary vale problems (2015), pp. [1-10]

Where $L(\delta, \omega)$ is holomorphic function defined by

$$L(\omega) = s + \sum_{\sigma=k+1}^{\infty} \frac{\omega}{\omega+\sigma+1} \xi_c s^{\sigma}. \text{ Note that form the above relations,}$$

$$(\beta^m(\omega, \beta, \lambda_c))' = \left(s + \sum_{c=k+1}^{\infty} \left(\frac{\omega}{\omega + \sigma - 1} \right) \frac{\beta^{c-1} e^{-\beta^2+1} \lambda_c}{(c-1)!} \xi_c s^c \right) * \left(\left(s + \sum_{\omega+c-1} \frac{\omega}{\omega + c - 1} \xi_c s^c \right) \right)' = \left(s + \sum_{\omega+c-1} \left(\frac{\omega}{\omega + c - 1} \right)^m \frac{\beta^{c-1} e^{-\beta^2+1} \lambda_c}{(c-1)!} \xi_c s^c \right)' = (\beta^m(\omega, \beta, \lambda_c))'.$$

$$\textbf{Theorem1.} \text{ Let } v = -y|\omega|\partial. \left(\partial > 0. \omega \ni \frac{\epsilon}{\{0\}}. y = \frac{\alpha_1 + \alpha_2}{2}. 0 < \alpha_1 + \alpha_2 \leq 1 \right) \quad (2.1)$$

Suppose that $f \in \mathcal{A}$ and

$$\mathcal{G}(s) = \left(\left(\frac{\beta^{m+1}(\omega, \beta, \lambda_e) F(s)}{\beta^m(\omega, \beta, \lambda_e) F(s)} \right) \left(\frac{\beta^m(\omega, \beta, \lambda_e) F(s)}{s} \right)^\partial \right)^m \geq 0 \text{ and } s \in \mathcal{U} \quad (2.2)$$

$$\text{If } \operatorname{Re} \left\{ 1 + \frac{\mathcal{G}(s)''}{\mathcal{G}(s)'} \right\} > v. (s \in \mathcal{U}) \quad (2.3)$$

And $k(s) \in \mathcal{A}$. Then the subordination condition

$$\left(\frac{\beta^{m+1}(\omega, \beta, \lambda_e) F(s)}{\beta^m(\omega, \beta, \lambda_e) F(s)} \right) \left(\frac{\beta^m(\omega, \beta, \lambda_e) F(s)}{s} \right)^\partial < \mathcal{G}(s) \quad (2.4)$$

Implies

$$\beta^m(\omega, \beta, \lambda_e) K(s) < \beta^m(\omega, \beta, \lambda_e) F(s) \quad (2.5)$$

Moreover, the function $\beta^m(\omega, \beta, \lambda_e) F(s)$ is the best dominant.

Proof. Let $I_1(s)$ and $I_2(s)$ are two holomorphic functions in $\mathcal{U}$, defined as

$$I_1(s) = \left(\frac{\beta^m(\omega, \beta, \lambda_e) K(s)}{s} \right)^\sigma$$

and

$$I_2(s) = \left(\frac{\beta^m(\omega, \beta, \lambda_e) F(s)}{s} \right)^\sigma \quad (2.6)$$

Suppose that, without loss of generality, that $I_2(s)$ is holomorphic and univalent in $\overline{\mathcal{U}}$

And $I_2(\xi) \neq 0$ ($|\xi| = 1$). If not, then for $0 < s < 1$, we have $I_1(s) = I_1(\xi s)$. These new functions have the desired properties on $\overline{\mathcal{U}}$, and we can use then in the proof of our result. Therefore, if we assume that $\xi \rightarrow 1$, we will get the result. So, if

$$Q(s) = 1 + \frac{s I_2''(s)}{I_2'(s)}, (s \in \mathcal{U}) \quad (2.7)$$

Then $\operatorname{Re}(Q(s)) > 0$ ($s \in \mathcal{U}$), since

$$\begin{aligned} I_2'(s) &= \sigma \left(\frac{\beta^m(\omega, \beta, \lambda_e) F(s)}{s} \right)^{\sigma-1} \left(\frac{s \left(\beta^m(\omega, \beta, \lambda_e) F(s) \right)' + \beta^m(\omega, \beta, \lambda_e) F(s)}{s^2} \right) \\ &= \sigma \left(\frac{\beta^m(\omega, \beta, \lambda_e) F(s)}{s} \right)^\sigma \left(\frac{\beta^m(\omega, \beta, \lambda_e) F(s)}{s} \right)^{-1} \left(\frac{s \left(\beta^m(\omega, \beta, \lambda_e) F(s) \right)' - \beta^m(\omega, \beta, \lambda_e) F(s)}{s^2} \right) \\ &= \sigma \left(\frac{\beta^m(\omega, \beta, \lambda_e) F(s)}{s} \right)^\sigma \left(\frac{s}{\beta^m(\omega, \beta, \lambda_e) F(s)} \right) \left(\frac{\frac{(1+\bar{\omega})}{e e^{-\beta^2+1}} \beta^m(\omega, \beta, \lambda_e) F(s) - \frac{(1+\bar{\omega})}{e e^{-\beta^2+1}} \beta^m(\omega, \beta, \lambda_e) F(s) - \beta^m(\omega, \beta, \lambda_e) F(s)}{s^2} \right) = \\ &= \sigma \left(\frac{\beta^m(\omega, \beta, \lambda_e) F(s)}{s} \right)^\sigma s^{-1} \left(\frac{\frac{(1+\bar{\omega})}{e e^{-\beta^2+1}} \beta^{m+1}(\omega, \beta, \lambda_e) F(s)}{\beta^m(\omega, \beta, \lambda_e) F(s)} - \frac{(1+\bar{\omega}) e^{-\beta^2+1}}{e e^{-\beta^2+1}} + 1 \right) = \\ &= \sigma \left(\frac{\beta^m(\omega, \beta, \lambda_e) F(s)}{s} \right)^\sigma s^{-1} \left(\frac{1+\bar{\omega}}{e e^{-\beta^2+1}} \frac{\beta^{m+1}(\omega, \beta, \lambda_e) F(s)}{\beta^m(\omega, \beta, \lambda_e) F(s)} - \frac{1+\bar{\omega} - e^{-\beta^2+1} + e^{-\beta^2+1}}{e e^{-\beta^2+1}} \right) \end{aligned}$$

$$I_2'(s) = \sigma \left(\frac{\beta^m(\omega, \beta, \lambda_e)F(s)}{s} \right)^\sigma \left(\frac{1+\bar{\omega}}{e^{e^{-\beta^2+1}}} \right) \left(\frac{\beta^{m+1}(\omega, \beta, \lambda_e)F(s)}{\beta^m(\omega, \beta, \lambda_e)F(s)} - 1 \right) \frac{e^{e^{-\beta^2+1}}}{1+\bar{\omega}}$$

$$\frac{e^{e^{-\beta^2+1}}}{1+\bar{\omega}} I_2'(s) = \sigma \left(\frac{\beta^m(\omega, \beta, \lambda_e)F(s)}{s} \right)^\sigma \left(\frac{\beta^{m+1}(\omega, \beta, \lambda_e)F(s)}{\beta^m(\omega, \beta, \lambda_e)F(s)} \right) - \sigma \left(\frac{\beta^m(\omega, \beta, \lambda_e)}{s} \right)^\sigma$$

$$= \sigma \mathcal{Y}(s) - \sigma I_2(s)$$

By simply computation, we get

$$\sigma \mathcal{Y}(s) = \sigma I_2(s) + \frac{e^{e^{-\beta^2+1}} s I_2(s)}{1+\bar{\omega}}$$

$$\mathcal{Y}(s) = I_2(s) + \frac{e^{e^{-\beta^2+1}} s I_2(s)}{1+\bar{\omega}} \quad (2.8)$$

Differentiate both sides of (2.8), we have

$$\mathcal{Y}'(s) = I_2'(s) + \frac{e^{e^{-\beta^2+1}} (s I_2''(s) + I_2'(s))}{\sigma(1+\bar{\omega})}$$

$$= I_2'(s) + \frac{s I_2''(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} + \frac{I_2'(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})}$$

$$= \left(1 + \frac{1}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} \right) I_2'(s) + \frac{s I_2''(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} \quad (2.9)$$

$$\text{Since } Q(s) = 1 + \frac{s I_2''(s)}{I_2'(s)}$$

$$Q(s) I_2'(s) = I_2'(s) + s I_2''(s)$$

$$s I_2''(s) = Q(s) I_2'(s) - I_2'(s)$$

That is

$$\mathcal{Y}'(s) = \left(1 + \frac{1}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} \right) I_2'(s) + \left[\frac{Q(s) I_2'(s) - I_2'(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} \right]$$

$$= \left[\left(1 + \frac{1}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} \right) + \frac{Q(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} - \frac{1}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} \right] I_2'(s)$$

$$= \left(1 + \frac{Q(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} \right) I_2'(s)$$

$$I_2'(s)$$

$$\ln \mathcal{Y}'(s) = \ln \left(\left(1 + \frac{Q(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} \right) I_2'(s) \right)$$

$$\frac{\mathcal{Y}''(s)}{\mathcal{Y}'(s)} = \frac{\left(1 + \frac{Q(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} \right) I_2''(s) + I_2'(s) \left(\frac{Q(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} \right)}{I_2'(s) \left(\frac{Q(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} \right)}$$

Therefore

$$1 + \frac{\mathcal{Y}''(s)}{\mathcal{Y}'(s)} = 1 + \frac{I_2''(s)}{I_2'(s)} + \frac{s Q(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega})} * \frac{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega}) + Q(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}}(1+\bar{\omega}) + Q(s)}$$

$$= 1 + \frac{I''_2(s)}{I'_2(s)} + \frac{sQ'(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})+Q(s)}} = k_1(s)$$

This is

$$k_1(s) = Q(s) + \frac{sQ'(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})+Q(s)}} \quad (2.10)$$

$$k_1(s) \frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})} = Q(s) + \frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})} + \frac{sQ'(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})+Q(s)}}$$

$$= \frac{\left(Q(s) + \frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})}\right)^2 + sQ'(s)}{\left(Q(s) + \frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})}\right)}$$

$$= Q(s) + \frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})} + \frac{sQ'(s)}{\frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})+Q(s)}}$$

So,

$$Re\left(k_1(s) \frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})}\right) > 0 \quad (s \in \mathcal{U}) \quad (2.11)$$

Next, and by using lemma 3, we conclude that the $Q(s)$ is a solution to the differential equation in (2.10) that lie in $\mathcal{H}(\mathcal{U})$. Such that $k_1(s) = Q(s) = 1$ at $s=0$.

Now, let

$$N(v_1, v_2) = v_1 + \frac{v_2}{v_1 + \frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})}} + v$$

Where v is given by (2.1), the relations (2.10) and (2.11), give us

$$Re\left(N(Q(s), sQ'(s))\right) > 0 \quad (s \in \mathcal{U})$$

Where this conditions implies that $Re(Q(s)) > 0$, so to prove this, we must show that $Re(N(iy, x)) \leq 0$ (where $y \in \mathbb{R}, x \leq -\frac{1+y^2}{2}$)

$$\begin{aligned} Re(N(iy, x)) &= Re\left\{iy + \frac{x}{iy + \frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})}} + v\right\} \\ &= \left\{ \frac{\frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})}x(1+\bar{\omega})}{y^2 + \left(\frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})}\right)^2} + v \right\} \\ &= \left\{ \frac{\frac{\sigma(1+y^2)}{2e^{e^{-\beta^2+1}}(1+\bar{\omega})}(1+\bar{\omega})}{y^2 + \left(\frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})}\right)^2} + v \right\} \\ &= \frac{-\left\{\left[(1+\bar{\omega}) - 2e^{e^{-\beta^2+1}}v\right]y^2 - 2v\frac{\sigma}{2e^{e^{-\beta^2+1}}(1+\bar{\omega})^2} + \sigma(1+\bar{\omega})\right\}}{2e^{e^{-\beta^2+1}}\left(y^2 + \left(\frac{\sigma}{e^{e^{-\beta^2+1}}(1+\bar{\omega})}\right)^2\right)} \end{aligned}$$

$$= \frac{-\psi(\alpha, \beta, \bar{\omega}, v, y)}{2e^{e^{-\beta^2+1}}(y^2 + ((1+\bar{\omega}))^2)}$$

Where

$$\psi(\alpha, \beta, \bar{\omega}, v, y) = \left[\left((1 + \bar{\omega}) - 2e^{e^{-\beta^2+1}}v \right) y^2 - 2v \frac{\sigma^2}{e^{e^{-\beta^2+1}}} (1 + \bar{\omega})^2 + \sigma(1 + \bar{\omega}) \right] \quad (2.13)$$

Clearly that $\psi(\alpha, \beta, \bar{\omega}, v, y)$ is positive for v given by (2.1). This implies that (2.12) holds, we conclude as an application of lemma (2) that $Re(Q(s)) > 0$. ($s \in \mathcal{U}$). also by (2.7) we know that $I_2(s)$ is convex, now we define a new function

$$\mathcal{T}(s, x) = I_2(s) + \frac{(1+x)e^{e^{-\beta^2+1}}(sI'_2(s))}{\sigma(1+\bar{\omega})} \quad (0 \leq x < \infty, \sigma > 0 \text{ and } \beta > 0, s \in \mathcal{U}) \quad (2.14)$$

By property of $I_2(s)$, we have

$$\left. \frac{\partial \mathcal{T}(s, x)}{\partial s} \right|_{s=0} = \left(1 + \frac{(1+x)e^{e^{-\beta^2+1}}}{\sigma(1+\bar{\omega})} \right) I'_2(0)$$

$$\neq 0 \quad (0 \leq x < \infty, \sigma > 0 \text{ and } \beta > 0, s \in \mathcal{U})$$

That is

$$\begin{aligned} \frac{s \frac{\partial \mathcal{T}(s, x)}{\partial s}}{\frac{\partial \mathcal{T}(s, x)}{\partial x}} &= \frac{s I'_2(s) + \frac{(1+x)e^{e^{-\beta^2+1}}(s^2 I''_2(s) + s I'_2(s))}{\sigma(1+\bar{\omega})}}{\frac{(1+x)e^{e^{-\beta^2+1}}}{\sigma(1+\bar{\omega})}} \\ &= \frac{\sigma(1+\bar{\omega}) s I'_2(s) + (1+x)e^{e^{-\beta^2+1}}(s^2 I''_2(s) + s I'_2(s))}{\sigma(1+\bar{\omega})} \frac{\sigma(1+\bar{\omega})}{e^{e^{-\beta^2+1}} s I'_2(s)} \\ &= \frac{\sigma(1+\bar{\omega})}{e^{e^{-\beta^2+1}}} + (1+x) \left(\frac{s^2 I''_2(s)}{I'_2(s)} \right) \\ &= \frac{\sigma(1+\bar{\omega})}{e^{e^{-\beta^2+1}}} + (1+x) Q(s) \end{aligned}$$

Hence

$$Re \left\{ \frac{s \frac{\partial \mathcal{T}(s, x)}{\partial s}}{\frac{\partial \mathcal{T}(s, x)}{\partial x}} \right\} = Re \left\{ \frac{\sigma(1+\bar{\omega})}{e^{e^{-\beta^2+1}}} + (1+x) Q(s) \right\} > 0. \quad (0 \leq x < \infty, \sigma > 0 \text{ and } \beta > 0, s \in \mathcal{U})$$

Thus, by applying lemma 1, we obtain that $\mathcal{T}(s, x)$ is a subordination chain, thus from the definition of subordination chain that

$$\mathcal{Y}(s) = I_2(s) + \frac{e^{e^{-\beta^2+1}} I'_2(s)}{\sigma(1+\bar{\omega})} = \mathcal{T}(s, 0) = I_2(s) + \frac{(1+0)e^{e^{-\beta^2+1}} I'_2(s)}{\sigma(1+\bar{\omega})}$$

Therefore, we get

$$\mathcal{T}(s, 0) < \mathcal{T}(s, x) \text{ for } 0 \leq x < \infty$$

This ensure that

$$\begin{aligned} \mathcal{T}(\mathcal{Y}, x) &\notin \mathcal{T}(\mathcal{U}, 0) = \mathcal{Y}(\mathcal{U}) \\ (0 \leq x < \infty, \mathcal{Y} \in \partial \mathcal{U}) \end{aligned} \quad (2.15)$$

If $I_1(s) \prec I_2(s)$.

As an application of lemma, there exist two points $s_0 \in \mathcal{U}$ and $y_0 \in \partial \mathcal{U}$ such that $I_1(s_0) = I_2(y_0)$ and $s_0 I'_1(s_0) = (1+x)y_0 I'_2(y_0)$ ($0 \leq x < \infty$)

$$(2.16)$$

Thus, by the relations (1.7) and (2.16), we have

$$\begin{aligned}
T(y_0) &= I_2(y_0) + \frac{(1+x)e^{-\beta^2+1}(y_0)}{\partial(1+\bar{\omega})} = I_1(s_0) + \frac{(1+x)e^{-\beta^2+1}s_0 I_1(s_0)}{\partial(1+\bar{\omega})} \\
&= \left(\frac{\beta^m(\omega, \beta, \lambda_c)}{s_0}\right)^\partial + \frac{e^{-\beta^2+1} \left(\frac{\beta^m(\omega, \beta, \lambda_c)k(s_0)}{s_0}\right)^{\partial-1} \left(\frac{s_0(\beta^m(\omega, \beta, \lambda_c)k(s_0))' - \beta^m(\omega, \beta, \lambda_c)k(s)}{s_0^2}\right)}{\partial(1+\bar{\omega})} \\
&= \left(\frac{\beta^m(\omega, \beta, \lambda_c)}{s_0}\right)^\partial + \frac{e^{-\beta^2+1} \left(\frac{\beta^m(\omega, \beta, \lambda_c)k(s_0)}{s_0}\right)^\partial s_0 \frac{s_0}{s_0^2 \beta^m(\omega, \beta, \lambda_c)k(s)} \left(\frac{s_0(1+\bar{\omega})(\beta^{m+1}(\omega, \beta, \lambda_c)k(s)) - (1+\bar{\omega} + e^{-\beta^2+1})\beta^m(\omega, \beta, \lambda_c)k(s)}{e^{-\beta^2+1}}\right)}{\partial(1+\bar{\omega})} \beta^m(\omega, \beta, \lambda_c)k(s) \\
&= \left(\frac{\beta^m(\omega, \beta, \lambda_c)k(s_0)}{s_0}\right)^\partial + \frac{\partial \left(\frac{\beta^m(\omega, \beta, \lambda_c)k(s_0)}{s_0}\right)^\partial \frac{1}{\beta^m(\omega, \beta, \lambda_c)k(s_0)} ((1+\bar{\omega})\beta^{m+1}(\omega, \beta, \lambda_c)k(s) - (1+\bar{\omega})\beta^m(\omega, \beta, \lambda_c)k(s))}{\partial(1+\bar{\omega})} \\
&= \left(\frac{\beta^{m+1}(\omega, \beta, \lambda_c)k(s)}{\beta^m(\omega, \beta, \lambda_c)k(s_0)}\right) \left(\frac{\beta^m(\omega, \beta, \lambda_c)k(s_0)}{s_0}\right)^\partial \in \mathcal{Y}(\mathcal{U})
\end{aligned}$$

This contradict with (2.15), thus we obtain that $I_1(s) < I_2(s)$.

Conclude that $I_1(s) = I_2(s)$, observing that $I_2(s)$ is the best dominant.

Thus, we get the required result.

The following result is an application of super ordination to univalent function.

Theorem 2. Suppose that $I_1(s), I_2(s)$ are two function in $\mathcal{A}$

$$\operatorname{Re}\left\{1 + \frac{sG''(s)}{G'(s)}\right\} > v \left(G'(s) = \left(\frac{\beta^{m+1}(\omega, \beta, \lambda_c)F(s)}{\beta^m(\omega, \beta, \lambda_c)F(s)}\right) \left(\frac{\beta^m(\omega, \beta, \lambda_c)F(s)}{s}\right)^\partial, m \geq 0, s \in \mathcal{U}\right) \quad (2.17)$$

When v is given by (2.1), suppose that $\frac{\beta^m(\omega, \beta, \lambda_c)F(s)}{s} \in Q$ and $\left(\frac{\beta^{m+1}(\omega, \beta, \lambda_c)k(s)}{\beta^m(\omega, \beta, \lambda_c)k(s_0)}\right) \left(\frac{\beta^m(\omega, \beta, \lambda_c)k(s)}{s}\right)^\partial$ is univalent in $\mathcal{U}$.

$$\text{Then } \left(\frac{\beta^{m+1}(\omega, \beta, \lambda_c)F(s)}{\beta^m(\omega, \beta, \lambda_c)F(s)}\right) \left(\frac{\beta^m(\omega, \beta, \lambda_c)F(s)}{s}\right)^\partial < \left(\frac{\beta^{m+1}(\omega, \beta, \lambda_c)k(s)}{\beta^m(\omega, \beta, \lambda_c)k(s_0)}\right) \left(\frac{\beta^m(\omega, \beta, \lambda_c)k(s)}{s}\right)^\partial$$

Ensure that

$$\left(\frac{\beta^m(\omega, \beta, \lambda_c)F(s)}{s}\right)^\partial < \left(\frac{\beta^m(\omega, \beta, \lambda_c)k(s)}{s}\right)^\partial$$

Thus the $\left(\frac{\beta^m(\omega, \beta, \lambda_c)F(s)}{s}\right)^\partial$ the best dominant.

The following result is a composite of theorem 1 and 2, which known sandwich theorem.

Theorem 3. Assume that $k(s), f_k(s) \in \mathcal{A}$. ($k = 1, 2$) and

$$\operatorname{Re}\left\{1 + \frac{sG''(s)}{G'(s)}\right\} > v \left(G'(s) = \left(\frac{\beta^{m+1}(\omega, \beta, \lambda_c)F_k(s)}{\beta^m(\omega, \beta, \lambda_c)F_k(s)}\right) \left(\frac{\beta^m(\omega, \beta, \lambda_c)F(s)}{s}\right)^\partial, k = 1, 2; \partial > 0; m \geq 0, s \in \mathcal{U}\right)$$

Where v given by (2.1). If $\frac{\beta^m(\omega, \beta, \lambda_c)F(s)}{s} \in Q$ and

$$k^*(s) = \left(\frac{\beta^{m+1}(\omega, \beta, \lambda_c)k(s)}{\beta^m(\omega, \beta, \lambda_c)k(s)}\right) \left(\frac{\beta^m(\omega, \beta, \lambda_c)k(s)}{s}\right)^\partial \text{ is univalent in } \mathcal{U}, \text{ then } G_1(s) G_1(s) < k^*(s) < G_2(s).$$

Implies that

$$\left(\frac{\beta^m(\omega, \beta, \lambda_c)F_1(s)}{s}\right)^\partial < \left(\frac{\beta^m(\omega, \beta, \lambda_c)k(s)}{s}\right)^\partial < \left(\frac{\beta^m(\omega, \beta, \lambda_c)F_2(s)}{s}\right)^\partial \text{ and } \left(\frac{\beta^m(\omega, \beta, \lambda_c)F_1(s)}{s}\right)^\partial \text{ represent the best subordinate and } \left(\frac{\beta^m(\omega, \beta, \lambda_c)F_2(s)}{s}\right)^\partial \text{ refer to the best dominant.}$$

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