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Moving Distributed Loads on A Non-Uniform Prestressed Bernoulli–Euler Cantilever Supported by A Pasternak Foundation

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Abstract
This work presents an analytical investigation of the dynamic response of a prestressed cantilever Bernoulli-Euler beam with non-uniform properties resting on a bi-parametric subgrade subjected to moving uniformly distributed loads. The cantilever configuration, with fixed-free boundary conditions, introduces asymmetry in the dynamic behaviour and is of practical importance in bridge overhangs and runway structures. The axial prestress is modelled as a spatially varying function along the span, and the beam rests on a foundation characterized by Winkler and Pasternak shear parameters. The fourth-order partial differential equation of motion is formulated and reduced using the Galerkin method employing cantilever beam mode shapes as trial functions. The resulting variable-coefficient second-order ODE is treated with the modified asymptotic method of Strubble to obtain an approximate decoupled system, which is subsequently solved using Laplace and other integral transformation techniques. Results are depicted in graphs showing deflection time histories and maximum dynamic deflections at the free end. The study shows that for cantilever beams, the dynamic response is dominated by free-end vibration, and that foundation parameters and variable prestress have a more pronounced stabilizing effect compared to simply supported cases, effectively suppressing flutter-like deflections under high-velocity moving loads.

Keywords: non-uniform beam, Bernoulli–Euler beam, moving distributed load, moving mass, prestress, Pasternak foundation, dynamic response, boundary conditions

1. Introduction

Moving loads on beams are central to the dynamic assessment of bridges, rails, machine members and other slender structures. A moving load may be idealised as a prescribed force, or as a moving mass whose inertia is coupled to the beam acceleration. The latter model is more involved but is physically closer to a vehicle or payload traversing a flexible member.

This paper develops the cantilever specialisation of a general model for a non-uniform prestressed Bernoulli–Euler beam resting on a two-parameter Pasternak foundation. The study considers both moving uniformly distributed force and moving uniformly distributed mass. The analysis retains the variation of flexural rigidity, mass per unit length and axial force along the span.

The cantilever case addresses the strongly asymmetric boundary condition encountered in projecting structures, crane members, turbine components and instrumented beams.

The paper is organised as follows. Section 2 states the mathematical model and assumptions. Section 3 describes the Galerkin reduction and asymptotic solution strategy. Section 4 gives the boundary conditions and resonance interpretation. Section 5 presents the numerical results and discussion. Section 6 concludes the paper and identifies checks required before submission.

2. Mathematical Model

Let x denote the spanwise coordinate, t time and w(x,t) the transverse displacement. The governing equation used in the thesis is

$$\partial^2/\partial x^2[EJ(x)\partial^2 w/\partial x^2] - \partial/\partial x[N(x)\partial w/\partial x] + \mu(x)\partial^2 w/\partial t^2 + K_0 w - G\partial^2 w/\partial x^2 = P(x,t).$$

Here E is Young's modulus, J(x) the variable second moment of area, μ(x) the mass per unit length, N(x) the axial-force field, K_0 the Winkler-type foundation stiffness and G the Pasternak shear-layer modulus.
The non-uniform fields are represented by the thesis choices

$$N(x)=N_0[1+\sin(\pi x/L)],\ J(x)=J_0[1+\sin(\pi x/L)]^3$$

and

$$\mu(x)=\mu_0[1+\sin(\pi x/L)].$$

The moving distributed force is represented through $P_F(x,t)=MgH(x-ct)$, where c is the load speed and H is the Heaviside function. For the moving-mass problem, the load includes the convective acceleration operator

$$\partial^2/\partial t^2 + 2c\partial^2/(\partial x\partial t) + c^2\partial^2/\partial x^2.$$

The initial condition is zero displacement and zero initial velocity. The load acts over $0 \le ct \le L$. The Pasternak foundation reaction is

$$K_0 w - G\partial^2 w/\partial x^2.$$

3. Solution Procedure

The variable-coefficient partial differential equation is reduced by a Galerkin expansion, $w(x,t)=\Sigma_m Y_m(t)X_m(x)$, where the spatial functions satisfy the selected boundary conditions. Substitution and weighted integration over $0 \le x \le L$ produce a coupled sequence of second-order ordinary differential equations in the modal amplitudes $Y_m(t)$.
The moving-force problem is obtained by neglecting the inertia of the travelling load. The moving-mass problem retains local, Coriolis and centripetal contributions through the convective acceleration. The resulting variable-coefficient modal equations are treated using the modified Struble asymptotic method and solved with Laplace-transform and convolution operations, following the thesis methodology.
For the cantilever configuration, the coefficients multiplying the admissible spatial functions are obtained by imposing the corresponding end conditions before the Galerkin projection. This is the principal mathematical distinction between the three companion papers.

4. Boundary Conditions and Resonance

At the clamped end, $w=0$ and $\partial w/\partial x=0$. At the free end, the bending moment and shear force vanish, represented by $\partial^2 w/\partial x^2=0$ and $\partial^3 w/\partial x^3=0$ in the beam model.
In the undamped model, resonance occurs when the frequency associated with the travelling load coincides with a natural frequency of the beam–foundation system. The moving-mass formulation modifies the effective modal coefficients and therefore shifts the resonance condition relative to the moving-force approximation.

5. Numerical Results and Discussion

The thesis numerical illustration uses L=12.192 m, $E=2.10924\times10^9$ N/m², $J_0=2.87698\times10^{-3}$ m⁴ and $\mu_0=2758.291$ kg/m. The response is plotted against time while varying the axial-force parameter N, the Pasternak shear modulus G and the foundation modulus K. Unless otherwise indicated, the plots use the parameter combinations reported in the thesis captions, including G=90,000, K=40,000, K=90,000 and N=20,000 or 200,000.
For the cantilever configuration, increasing axial force, shear-layer modulus and foundation stiffness suppresses the predicted transverse response. The moving-mass solution can exceed the moving-force solution in the comparative case, demonstrating that load inertia is important for a reliable dynamic assessment of restrained–free members.

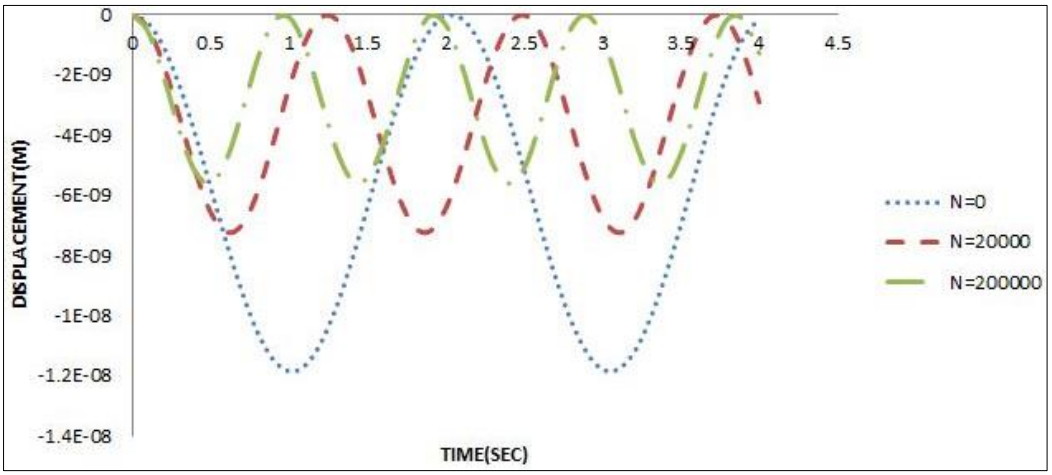


Fig 1: Effect of axial force on the moving-force response. Reproduced from the thesis numerical results; verify axis labels, units and image quality against the original plotting files before submission.

5.1. Interpretation of Parameter Effects

Across the parameter sweeps, increasing axial force reduces the response amplitude. Increasing the Pasternak shear-layer modulus also suppresses the transverse response by increasing the foundation's resistance to curvature-related deformation. Increasing the foundation stiffness K produces a further reduction in amplitude. These trends are consistent for both load idealisations, although the magnitude and phase of the response depend on whether load inertia is included.
The moving-force and moving-mass comparisons are not interchangeable. The thesis reports a larger moving-force response than the moving-mass response for the simply

supported and clamped–clamped comparison cases, while the cantilever comparison reports the reverse ordering. This boundary-condition dependence is a central reason to publish the three studies separately rather than combine them into a single undifferentiated paper.

6. Conclusions

A mathematical and numerical study has been presented for a non-uniform prestressed Bernoulli–Euler beam on a Pasternak foundation under moving uniformly distributed force and moving uniformly distributed mass, specialised to the cantilever boundary condition.

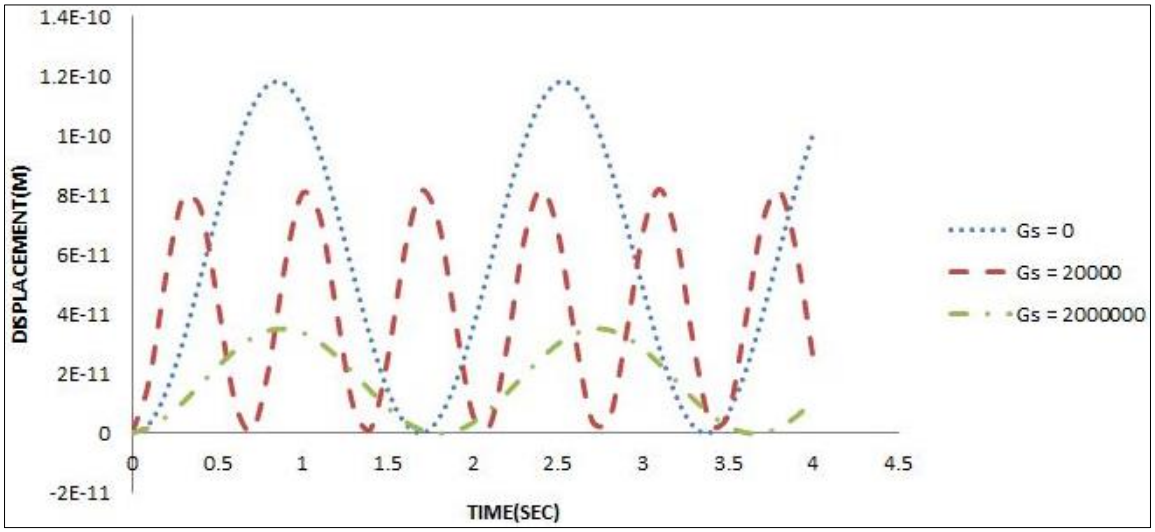


Fig 2: Effect of Pasternak shear modulus on the moving-force response. Reproduced from the thesis numerical results; verify axis labels, units and image quality against the original plotting files before submission.

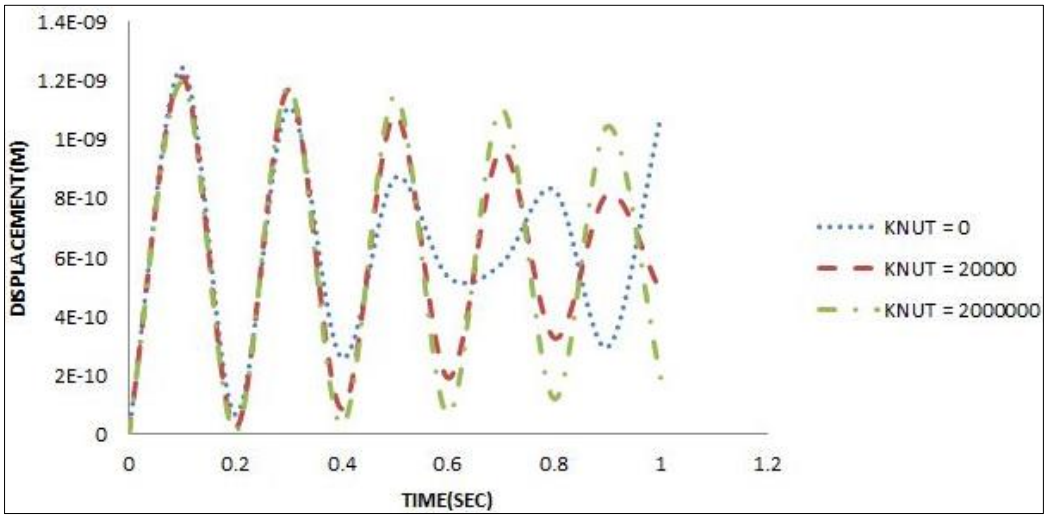


Fig 3: Effect of foundation stiffness on the moving-force response. Reproduced from the thesis numerical results; verify axis labels, units and image quality against the original plotting files before submission.

The principal conclusions are:

- The variable axial-force parameter suppresses the response amplitude in the reported numerical cases.
- The Pasternak shear modulus and foundation stiffness reduce the dynamic transverse response.
- Moving-force and moving-mass models can produce different amplitudes and phase histories; the moving-force approximation is not universally conservative.
- Boundary conditions change the response ordering and therefore deserve separate design interpretations.

Before journal submission, the numerical results should be regenerated from the original computational implementation, the dimensional consistency of all coefficients should be checked, and the figure labels should be corrected and standardised. The thesis text contains typographical inconsistencies in symbols and some figure descriptions; these have been retained only where needed to preserve traceability and should not be treated as final publication copy.

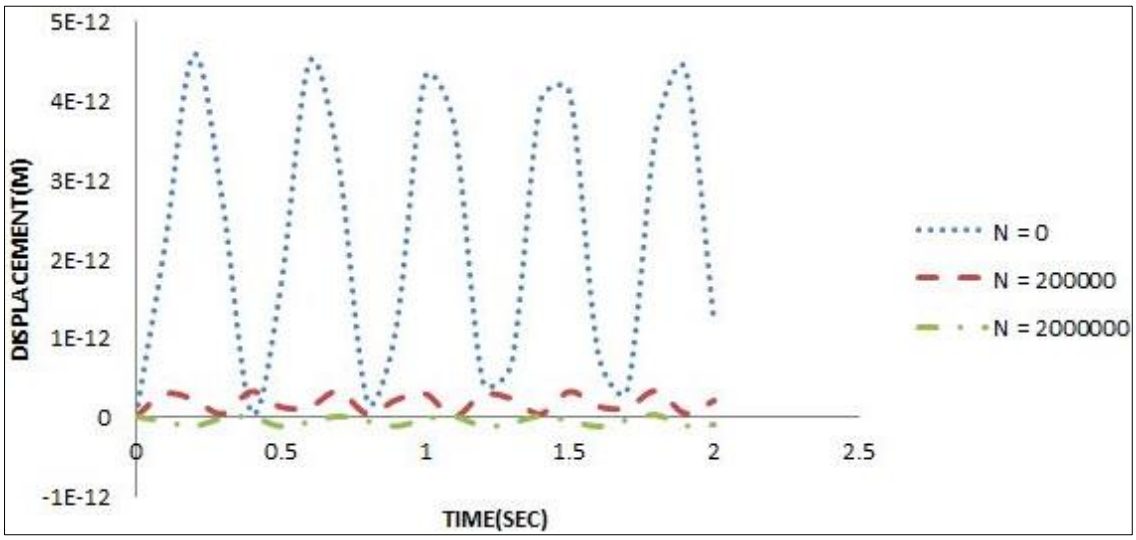


Fig 4: Effect of axial force on the moving-mass response. Reproduced from the thesis numerical results; verify axis labels, units and image quality against the original plotting files before submission.

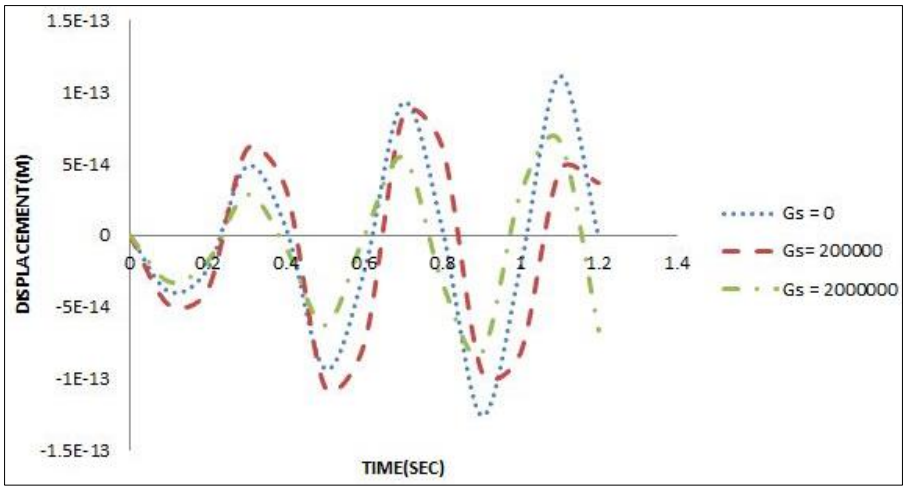


Fig 5: Effect of Pasternak shear modulus on the moving-mass response. Reproduced from the thesis numerical results; verify axis labels, units and image quality against the original plotting files before submission.

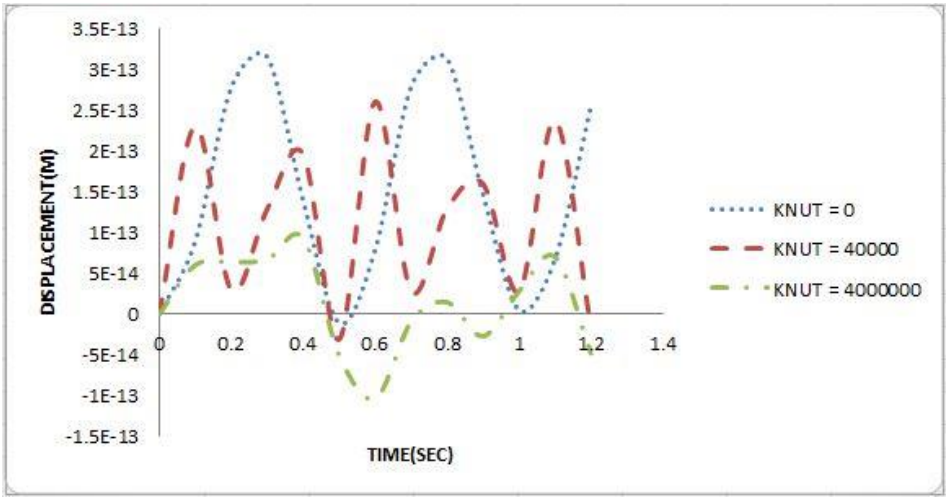


Fig 6: Effect of foundation stiffness on the moving-mass response. Reproduced from the thesis numerical results; verify axis labels, units and image quality against the original plotting files before submission.

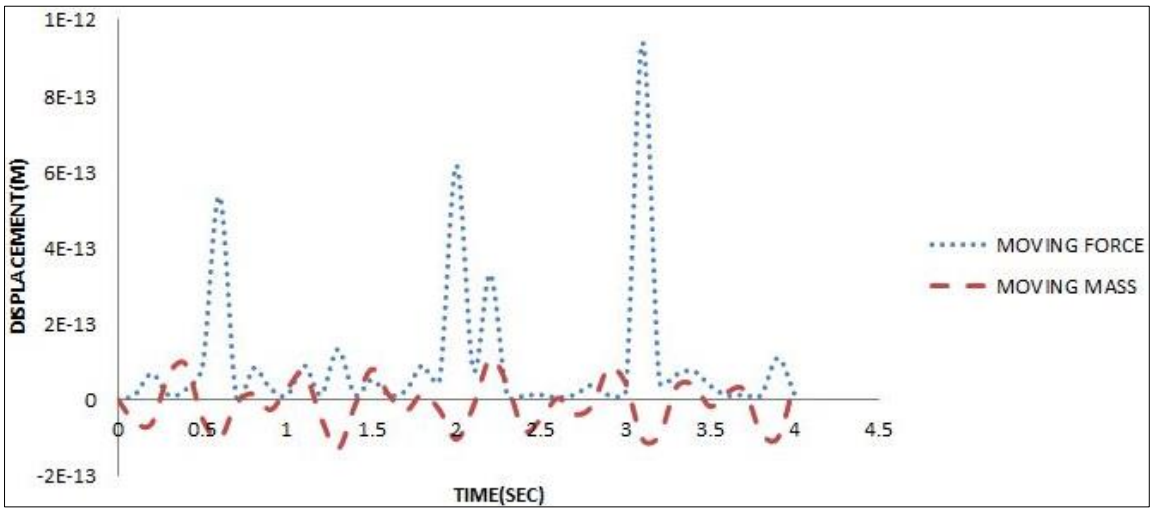


Fig 7: Comparison of moving-force and moving-mass responses. Reproduced from the thesis numerical results; verify axis labels, units and image quality against the original plotting files before submission.

Declarations

Funding: Not stated in the source thesis. Conflict of interest: The authors should confirm whether any conflict exists. Data availability: The numerical data underlying the figures should be deposited or made available on reasonable request. Author contributions: Ebenezer Ajomole developed the thesis analysis and manuscript draft; Prof. S. T. Oni supervised the work.

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